

New Optional Stopping Theorems and Maximal Inequalities on Stochastic Processes ^{*}

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Abstract

In this paper, we develop new optional stopping theorems for scenarios where the stopping rules are defined by bounded continuity regions. Moreover, we establish a wide variety of inequalities on the supremums and infimums of functions of stochastic processes over the whole range of time indexes.

1 Introduction

Martingale theory has been developed as a powerful tool for investigating stochastic processes. One of the most important results of martingale theory are Doob's optional stopping theorem [8] and its variations. These optional stopping theorems are relied on the assumptions such as uniform integrability or integrable stopping times. However, in many applications, the relevant stochastic process is not uniformly integrable and the expectation of the stopping time is not necessarily finite. Motivated by this situation, in this paper, we shall develop new optional stopping theorems for scenarios where the uniform integrability of the stochastic process or the integrability of the stopping time are not guaranteed, while the continuity region associated with the stopping rule is bounded. Based on the new optional stopping theorems, we have established general maximal inequalities, which accommodate some classical inequalities such as, Bernstein's inequality [3], Chernoff bounds [4, 5], Bennett's inequality [2], Hoeffding-Azuma inequality [1, 9] as special cases.

This paper is organized as follows. In Section 2, we present new optional stopping theorems. In Section 3, we propose new maximal inequalities. Section 3 is the conclusion. All proofs are given in the Appendices. The main results of this paper have appeared in [7].

Throughout this paper, we shall use the following notations. Let \mathbb{R} denote the set of real numbers. Let \mathbb{R}^+ denote the set of non-negative real numbers. Let \mathbb{Z}^+ denote the set of non-negative integers. Let \mathbb{N} denote the set of positive integers. Let $(X_t)_{t \in \mathbb{T}}$ denote a stochastic process, where $\mathbb{T} \subseteq \mathbb{R}^+$ is the set of time values. Specially, $(X_t)_{t \in \mathbb{T}}$ is a continuous-time stochastic process if $\mathbb{T} = \mathbb{R}^+$; and $(X_t)_{t \in \mathbb{T}}$ is a discrete-time stochastic process if $\mathbb{T} = \mathbb{Z}^+$. We assume that all stochastic processes are defined in probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We also use \mathbb{P} to denote the probability measure \Pr . For $t \in \mathbb{T}$, let \mathcal{F}_t denote the sub- σ -algebra generated by the collection of random variables $\{X_\tau : 0 \leq \tau \leq t, \tau \in \mathbb{T}\}$. The collection $(\mathcal{F}_t)_{t \in \mathbb{T}}$ of sub- σ -algebras of \mathcal{F} is called the natural filtration of \mathcal{F} . Let " $A \vee B$ " denote the maximum of A and B . Let " $A \wedge B$ " denote the minimum of A and B . The other notations will be made clear as we proceed.

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2 Optional Stopping Theorems

In this section, we shall first develop some new optional stopping theorems on stochastic processes. Consider a stochastic process $(X_t)_{t \in \mathbb{R}^+}$ defined in the probability space $(\Omega, \mathcal{F}, \Pr)$. Let τ be a stopping time taking values in $\mathbb{R}^+ \cup \{\infty\}$. Define

$$X_\tau = \lim_{t \rightarrow \infty} X_{\tau \wedge t}$$

if the limit exists. Clearly, for $\omega \in \Omega$,

$$X_\tau(\omega) = \begin{cases} X_{\tau(\omega)}(\omega) & \text{if } \tau(\omega) < \infty; \\ \lim_{t \rightarrow \infty} X_t(\omega) & \text{if } \tau(\omega) = \infty \text{ and the limit exists.} \end{cases}$$

Let τ_1 and τ_2 be two stopping times. Since the stopping times can be ∞ , we shall define the notion of $\tau_1 \leq \tau_2$ as follows:

$$\begin{aligned} \{\tau_1 \leq \tau_2\} &= \{\omega \in \Omega : \tau_1(\omega) \leq \tau_2(\omega), \tau_1(\omega) \in \mathbb{R}^+, \tau_2(\omega) \in \mathbb{R}^+\} \\ &\cup \{\omega \in \Omega : \tau_1(\omega) \in \mathbb{R}^+, \tau_2(\omega) = \infty\} \cup \{\omega \in \Omega : \tau_1(\omega) = \infty, \tau_2(\omega) = \infty\}. \end{aligned}$$

Clearly, a discrete-time process $(X_k)_{k \in \mathbb{Z}^+}$ can be viewed as a right-continuous process $(X_t)_{t \in \mathbb{R}^+}$ with $X_t = X_k$ for $t \in [k, k+1)$, $k \in \mathbb{Z}^+$. Therefore, we shall consider the optional stopping problems in the general setting of continuous-time processes. However, in order to develop new optional stopping theorems for continuous-time processes, we first need to establish discrete-time optional stopping theorems and then generalize them to continuous-time processes. For a discrete-time process, we have the following general results.

Theorem 1 *Let $(X_k, \mathcal{F}_k)_{k \in \mathbb{Z}^+}$ be a discrete-time super-martingale. Let τ_1 and τ_2 be two stopping times such that $\tau_1 \leq \tau_2$ almost surely and that there exists a constant C so that $\{\tau_2 > k\} \subseteq \{|X_k| < C\}$ for all $k \in \mathbb{Z}^+$. Assume that*

$$X_{\tau_2} \text{ exist and } \mathbb{E}[|X_{\tau_2}|] \text{ is finite.} \quad (1)$$

Then, $\mathbb{E}[X_{\tau_2} | \mathcal{F}_{\tau_1}] \leq X_{\tau_1}$ and $\mathbb{E}[X_{\tau_2}] \leq \mathbb{E}[X_{\tau_1}]$ almost surely, with equality if $(X_k, \mathcal{F}_k)_{k \in \mathbb{Z}^+}$ is a martingale. Specially, the assumption (1) is satisfied and the conclusion follows in the following cases:

- (i) *$(X_k, \mathcal{F}_k)_{k \in \mathbb{Z}^+}$ is a super-martingale such that there exists a constant Δ so that $|X_{k+1} - X_k| < \Delta$ almost surely for all $k \in \mathbb{Z}^+$.*
- (ii) *$(X_k, \mathcal{F}_k)_{k \in \mathbb{Z}^+}$ is a non-negative super-martingale.*

See Appendix A for a proof.

In particular, as an immediate application of Theorem 1, we have the following result.

Corollary 1 *Let $(X_k)_{k \in \mathbb{Z}^+}$ be a discrete-time stochastic process such that $X_0, X_n - X_{n-1}$, $n = 1, 2, \dots$ are independent random variables with zero means and that $\mathbb{E}[|X_0|] < \infty$, $\sup_{n \geq 0} |X_n - X_{n-1}| < \infty$ almost surely. Let τ_1 and τ_2 be two stopping times such that $\tau_1 \leq \tau_2$ almost surely and that there exists a constant C so that $\{\tau_2 > n\} \subseteq \{|X_n| < C\}$ for all $n \geq 0$. Then, $\mathbb{E}[X_{\tau_2} | \mathcal{F}_{\tau_1}] = X_{\tau_1}$ and $\mathbb{E}[X_{\tau_2}] = \mathbb{E}[X_{\tau_1}]$ almost surely.*

To investigate optional stopping problems for continuous-time processes, throughout the remainder of this paper, we shall define two collections of sets of real numbers, denoted by $\{\mathcal{D}_t, t \in \mathbb{R}^+\}$ and $\{\mathcal{D}_t, t \in \mathbb{R}^+\}$, such that the following requirements are satisfied:

- (i) $\mathcal{D}_t \subseteq \mathcal{D}_t$ for all $t \in \mathbb{R}^+$.
- (ii) There exists a positive constant C such that $\mathcal{D}_t \subseteq [-C, C]$ for all $t \in \mathbb{R}^+$.

(iii) For any right-continuous function $g(t) : \mathbb{R}^+ \mapsto \mathbb{R}$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \inf\{t \in \mathbb{S}_n : g(t) \notin \mathcal{D}_t\} &= \inf\{t \in \mathbb{R}^+ : g(t) \notin \mathcal{D}_t\}, \\ \lim_{n \rightarrow \infty} \inf\{t \in \mathbb{S}_n : g(t) \notin \mathcal{D}_t\} &= \inf\{t \in \mathbb{R}^+ : g(t) \notin \mathcal{D}_t\}, \end{aligned}$$

where $\mathbb{S}_n = \{k2^{-n} : k \in \mathbb{Z}^+\}$ for $n \in \mathbb{N}$. Note that the infimums can be ∞ .

Clearly, if a stopping time τ is defined such that $\{\tau > t\}$ implies $\{X_t \in \mathcal{D}_t\}$, then the region of (X_t, t) for continuing observing $(X_t)_{t \in \mathbb{R}^+}$ is bounded. In this sense, a stopping rule with such a stopping time is called a stopping rule with *bounded continuity region*.

In many areas of engineering and sciences, it is a frequent problem to investigate a stochastic process with bounded rate of variation. For this purpose, the following result is useful.

Theorem 2 *Let $(X_t, \mathcal{F}_t)_{t \in \mathbb{R}^+}$ be a right-continuous super-martingale such that there exist constants δ and $\Delta > 0$ so that $|X_{t'} - X_t| < \Delta$ almost surely provided that $|t' - t| \leq \delta$. Let $\tau_1 = \inf\{t \in \mathbb{R}^+ : X_t \notin \mathcal{D}_t\}$ and $\tau_2 = \inf\{t \in \mathbb{R}^+ : X_t \notin \mathcal{D}_t\}$. Then, $\mathbb{E}[X_{\tau_2} | \mathcal{F}_{\tau_1}] \leq X_{\tau_1}$ and $\mathbb{E}[X_{\tau_2}] \leq \mathbb{E}[X_{\tau_1}]$ almost surely, with equality if $(X_t, \mathcal{F}_t)_{t \in \mathbb{R}^+}$ is a martingale.*

See Appendix B for a proof.

Theorem 3 *Let $(X_t, \mathcal{F}_t)_{t \in \mathbb{R}^+}$ be a right-continuous, non-negative super-martingale. Let $\tau_1 = \inf\{t \in \mathbb{R}^+ : X_t \notin \mathcal{D}_t\}$ and $\tau_2 = \inf\{t \in \mathbb{R}^+ : X_t \notin \mathcal{D}_t\}$. Then, $\mathbb{E}[X_{\tau_2} | \mathcal{F}_{\tau_1}] \leq X_{\tau_1}$ and $\mathbb{E}[X_{\tau_2}] \leq \mathbb{E}[X_{\tau_1}]$ almost surely, with equality if $(X_t, \mathcal{F}_t)_{t \in \mathbb{R}^+}$ is a martingale.*

See Appendix C for a proof.

It should be noted that Theorem 3 can be readily generalized to a right-continuous super-martingale which is bounded from below by a constant. Making use of Theorem 3, we have established Corollary 2 as follows.

Corollary 2 *Let \mathcal{V}_t be a right-continuous function of $t \geq 0$. Let $(X_t)_{t \in \mathbb{R}^+}$ be a right-continuous stochastic process such that*

$$\mathbb{E}[\exp(sX_0)] \leq \exp(\varphi(s)\mathcal{V}_0), \quad \mathbb{E}[\exp(s(X_t - X_\tau)) | \mathcal{F}_\tau] \leq \exp(\varphi(s)(\mathcal{V}_t - \mathcal{V}_\tau)) \quad (2)$$

almost surely for arbitrary $t \geq \tau \geq 0$ and $s \in (-a, b)$, where a and b are positive numbers or infinity, and $\varphi(s)$ is a function of $s \in (-a, b)$. Define $Y_t = \exp(sX_t - \varphi(s)\mathcal{V}_t)$ for $s \in (-a, b)$. Let $\tau_1 = \inf\{t \in \mathbb{R}^+ : Y_t \notin \mathcal{D}_t\}$ and $\tau_2 = \inf\{t \in \mathbb{R}^+ : Y_t \notin \mathcal{D}_t\}$. Then, $\mathbb{E}[Y_{\tau_2} | \mathcal{F}_{\tau_1}] \leq Y_{\tau_1}$ and $\mathbb{E}[Y_{\tau_2}] \leq \mathbb{E}[Y_{\tau_1}]$ almost surely, with equality if (2) holds with equality almost surely for arbitrary $t \geq \tau \geq 0$ and $s \in (-a, b)$.

3 Maximal Inequalities

By virtue of the above optional stopping theorems, we shall establish some general maximal inequalities. With regard to a uniformly integrable (UI) martingale process, we have discovered the following fact.

Theorem 4 *If $(X_t, \mathcal{F}_t)_{t \in \mathbb{R}^+}$ is a right-continuous uniformly integrable martingale which converges almost surely to a constant c , then X_t is equal to the constant c for all $t \geq 0$ almost surely.*

See Appendix D for a proof.

Theorem 4 implies that a right-continuous non-constant UI martingale never converges to a constant. For a super-martingale converging to a constant, we have the following results.

Theorem 5 Let $(X_t, \mathcal{F}_t)_{t \in \mathbb{R}^+}$ be a right-continuous, non-negative super-martingale which converges almost surely to a constant c . Then, $\Pr \left\{ \sup_{t \geq 0} X_t \geq \gamma \right\} \leq \frac{\mathbb{E}[X_0] - c}{\gamma - c}$ for any $\gamma > c$. Specially, $\Pr \left\{ \sup_{t \geq 0} X_t \geq \gamma \right\}$ is equal to $\frac{\mathbb{E}[X_0] - c}{\gamma - c}$ and 1 in accordance with $\gamma > \mathbb{E}[X_0]$ and $\gamma \leq \mathbb{E}[X_0]$ under additional assumption that $(X_t, \mathcal{F}_t)_{t \in \mathbb{R}^+}$ is a continuous martingale.

See Appendix E for a proof.

By virtue of Corollary 2 and Markov's inequality, the following Corollary 3 can be established.

Corollary 3 Let \mathcal{V}_t be a right-continuous function of $t \geq 0$. Let $(X_t)_{t \in \mathbb{R}^+}$ be a right-continuous stochastic process such that $\mathbb{E}[\exp(sX_0)] \leq \exp(\varphi(s)\mathcal{V}_0)$ and $\mathbb{E}[\exp(s(X_t - X_\tau)) \mid \mathcal{F}_\tau] \leq \exp(\varphi(s)(\mathcal{V}_t - \mathcal{V}_\tau))$ almost surely for arbitrary $t \geq \tau \geq 0$ and $s \in (-a, b)$, where a and b are positive numbers or infinity, and $\varphi(s)$ is a function of $s \in (-a, b)$. Define $Y_t = \exp(sX_t - \varphi(s)\mathcal{V}_t)$ for $s \in (-a, b)$. Then, $\Pr \left\{ \sup_{t \geq 0} Y_t \geq \gamma \right\} \leq \frac{1}{\gamma}$ for $\gamma > 0$.

As a direct consequence of Theorem 5, we have shown the following Corollary 4.

Corollary 4 Let \mathcal{V}_t be a non-negative, continuous function of $t \geq 0$ such that the limit inferior of $\mathcal{V}_{n+1} - \mathcal{V}_n$ with respect to $n \in \mathbb{N}$ is positive. Let $(X_t)_{t \in \mathbb{R}^+}$ be a continuous stochastic process such that $\mathbb{E}[\exp(sX_0)] = \exp(\varphi(s)\mathcal{V}_0)$ and $\mathbb{E}[\exp(s(X_t - X_\tau)) \mid \mathcal{F}_\tau] = \exp(\varphi(s)(\mathcal{V}_t - \mathcal{V}_\tau))$ almost surely for arbitrary $t \geq \tau \geq 0$ and $s \in (-a, b)$, where a and b are positive numbers or infinity, and $\varphi(s)$ is a function of $s \in (-a, b)$. Define $Y_t = \exp(sX_t - \varphi(s)\mathcal{V}_t)$ for $s \in (-a, 0) \cup (0, b)$. Then, $\Pr \left\{ \sup_{t \geq 0} Y_t \geq \gamma \right\} = \frac{1}{\gamma}$ for $\gamma \geq 1$.

See Appendix F for a proof.

Making use of Corollary 3, we have developed the following results concerning stochastic processes.

Theorem 6 Let \mathcal{V}_t be a non-negative, right-continuous function of $t \in [0, \infty)$. Let $(X_t)_{t \in \mathbb{R}^+}$ be a right-continuous stochastic process such that $\mathbb{E}[\exp(s(X_{t'} - X_t)) \mid \mathcal{F}_t] \leq \exp((\mathcal{V}_{t'} - \mathcal{V}_t)\varphi(s))$ almost surely for arbitrary $t' \geq t \geq 0$ and $s \in (-a, b)$, where a and b are positive numbers or infinity, and $\varphi(s)$ is a non-negative function of $s \in (-a, b)$. Let $\tau \geq 0$, $\gamma > 0$, $\eta \geq 0$. Define $\mathcal{A} = \{s \in (0, a) : \varphi(-s) \leq \gamma s\}$ and $\mathcal{B} = \{s \in (0, b) : \varphi(s) \leq \gamma s\}$. Then,

$$\Pr \left\{ \inf_{t > 0} \left[X_t - X_0 + \gamma \mathcal{V}_\tau + \frac{\varphi(-s)}{s}(\mathcal{V}_t - \mathcal{V}_\tau) \right] \leq 0 \right\} \leq [\exp(\varphi(-s) - \gamma s)]^{\mathcal{V}_\tau} \quad \forall s \in (0, a), \quad (3)$$

$$\Pr \left\{ \sup_{t > 0} \left[X_t - X_0 - \gamma \mathcal{V}_\tau - \frac{\varphi(s)}{s}(\mathcal{V}_t - \mathcal{V}_\tau) \right] \geq 0 \right\} \leq [\exp(\varphi(s) - \gamma s)]^{\mathcal{V}_\tau} \quad \forall s \in (0, b), \quad (4)$$

$$\Pr \left\{ \inf_{t > 0} [X_t - X_0 + \gamma(\mathcal{V}_\tau \vee \mathcal{V}_t)] \leq 0 \right\} \leq \inf_{s \in (0, a)} [\exp(\varphi(-s) - \gamma s)]^{\mathcal{V}_\tau}, \quad (5)$$

$$\Pr \left\{ \sup_{t > 0} [X_t - X_0 - \gamma(\mathcal{V}_\tau \vee \mathcal{V}_t)] \geq 0 \right\} \leq \inf_{s \in (0, b)} [\exp(\varphi(s) - \gamma s)]^{\mathcal{V}_\tau} \quad \text{and moreover,} \quad (6)$$

$$\Pr \left\{ \inf_{t > 0} (X_t - X_0 + \eta + \gamma \mathcal{V}_t) \leq 0 \right\} \leq \inf_{s \in \mathcal{A}} e^{-\eta s}, \quad (7)$$

$$\Pr \left\{ \sup_{t > 0} (X_t - X_0 - \eta - \gamma \mathcal{V}_t) \geq 0 \right\} \leq \inf_{s \in \mathcal{B}} e^{-\eta s}, \quad (8)$$

$$\Pr \left\{ \inf_{t > 0} [X_t - X_0 + \eta + \gamma(\mathcal{V}_\tau \vee \mathcal{V}_t)] \leq 0 \right\} \leq \inf_{s \in \mathcal{A}} e^{-\eta s} [\exp(\varphi(-s) - \gamma s)]^{\mathcal{V}_\tau}, \quad (9)$$

$$\Pr \left\{ \sup_{t > 0} [X_t - X_0 - \eta - \gamma(\mathcal{V}_\tau \vee \mathcal{V}_t)] \geq 0 \right\} \leq \inf_{s \in \mathcal{B}} e^{-\eta s} [\exp(\varphi(s) - \gamma s)]^{\mathcal{V}_\tau} \quad (10)$$

provided that \mathcal{A} and \mathcal{B} are nonempty respectively. In particular, under the above assumptions on X_t , \mathcal{V}_t and $\varphi(s)$, the following statements hold true:

(I): If $\varphi(s)$ is a continuous function smaller than $\gamma|s|$ at a neighborhood of 0, then

$$\Pr\{\inf_{t>0}[X_t - X_0 + \gamma\mathcal{V}_t + \alpha(\gamma)(\mathcal{V}_t - \mathcal{V}_\tau)] \leq 0\} \leq \inf_{s \in (0,a)} [\exp(\varphi(-s) - \gamma s)]^{\mathcal{V}_\tau}, \quad (11)$$

$$\Pr\{\sup_{t>0}[X_t - X_0 - \gamma\mathcal{V}_t - \beta(\gamma)(\mathcal{V}_t - \mathcal{V}_\tau)] \geq 0\} \leq \inf_{s \in (0,b)} [\exp(\varphi(s) - \gamma s)]^{\mathcal{V}_\tau}, \quad (12)$$

where $\alpha(\gamma)$ and $\beta(\gamma)$ are functions of γ defined as follows: $\alpha(\gamma)$ is equal to $\frac{\varphi(-s^*)}{s^*}$ if $\inf_{s \in (0,a)} [\varphi(-s) - \gamma s]$ is attained at $s^* \in (0,a)$ and otherwise equal to $\lim_{s \uparrow a} \frac{\varphi(-s)}{s}$; $\beta(\gamma)$ is equal to $\frac{\varphi(s^*)}{s^*}$ if $\inf_{s \in (0,b)} [\varphi(s) - \gamma s]$ is attained at $s^* \in (0,b)$ and otherwise equal to $\lim_{s \uparrow b} \frac{\varphi(s)}{s}$. Moreover, $0 < \alpha(\gamma) < \gamma$ and $0 < \beta(\gamma) < \gamma$.

(II): If $\frac{\varphi(s)}{|s|}$ is monotonically increasing with respect to $|s| > 0$, then

$$\Pr\{\inf_{t>0}[X_t - X_0 + \eta + \gamma(\mathcal{V}_t \vee \mathcal{V}_\tau)] \leq 0\} \leq \inf_{s \in (0,a^*)} e^{-\eta s} [\exp(\varphi(-s) - \gamma s)]^{\mathcal{V}_\tau} \quad (13)$$

and

$$\Pr\{\sup_{t>0}[X_t - X_0 - \eta - \gamma(\mathcal{V}_t \vee \mathcal{V}_\tau)] \geq 0\} \leq \inf_{s \in (0,b^*)} e^{-\eta s} [\exp(\varphi(s) - \gamma s)]^{\mathcal{V}_\tau}, \quad (14)$$

where a^* and b^* are defined as follows: a^* is equal to a if $\lim_{s \uparrow a} \frac{\varphi(-s)}{s} \leq \gamma$ and otherwise equal to $s \in (0,a)$ such that $\frac{\varphi(-s)}{s} = \gamma$; b^* is equal to b if $\lim_{s \uparrow b} \frac{\varphi(s)}{s} \leq \gamma$ and otherwise equal to $s \in (0,b)$ such that $\frac{\varphi(s)}{s} = \gamma$.

See Appendix G for a proof.

An important application of Theorem 6 is illustrated as follows. Let Y be a random variable with mean μ . Define $X_0 = 0$ and $X_n = \sum_{i=1}^n (Y_i - \mu)$ for $n \in \mathbb{N}$, where Y_1, Y_2, \dots are i.i.d. samples of Y . Define a right-continuous stochastic process $(X_t)_{t \in \mathbb{R}^+}$ such that $X_t = X_n$ for $t \in [n, n+1)$, $n = 0, 1, 2, \dots$. Define a right-continuous function \mathcal{V}_t of $t \in \mathbb{R}^+$ such that $\mathcal{V}_t = n$ for $t \in [n, n+1)$, $n = 0, 1, 2, \dots$. If there is a convex function $\varphi(s)$ such that $\ln \mathbb{E}[\exp(s(Y - \mu))] \leq \varphi(s)$ for $s \in (-a, b)$ and that $\varphi(0) = 0$, then we can apply Theorem 6 to develop maximal inequalities for $(X_t)_{t \in \mathbb{R}^+}$, which immediately lead to maximal inequalities for $(X_n)_{n \in \mathbb{N}}$. The function $\varphi(s)$ of the desired properties can be found for some particular cases as follows:

- (i) If the moment generating function of Y exists, then $\varphi(s)$ can be taken as $\ln \mathbb{E}[\exp(sY)] - \mu s$.
- (ii) If Y is a random variable such that $\mathbb{E}[Y] = 0$, $\mathbb{E}[Y^2] = \sigma^2$ and $Y \leq b$, then by Bennett's inequality [2], the function $\varphi(s)$ can be taken as $\ln[\frac{b^2}{b^2 + \sigma^2} \exp(-\frac{\sigma^2}{b} s) + \frac{\sigma^2}{b^2 + \sigma^2} \exp(bs)]$.
- (iii) If Y is a random variable bounded in interval $[0, 1]$ almost surely, then by Hoeffding's inequality [9], the function $\varphi(s)$ can be taken as $\ln(1 - \mu + \mu e^s) - \mu s$.
- (iv) If Y is a random variable uniformly distributed over $[-\frac{1}{2}, \frac{1}{2}]$, then we can show that $\mathbb{E}[e^{sY}] \leq \exp(\frac{s^2}{24})$ for all real number s . Hence, the function $\varphi(s)$ can be taken as $\frac{s^2}{24}$. See Appendix H for the development of the bound for the moment generating function $\mathbb{E}[e^{sY}]$.

Applying Corollary 4 to a continuous stochastic process, we have the following results.

Theorem 7 Let \mathcal{V}_t be a non-negative, continuous function of $t \in [0, \infty)$ such that the limit inferior of $\mathcal{V}_{n+1} - \mathcal{V}_n$ with respect to $n \in \mathbb{N}$ is positive. Let $(X_t)_{t \in \mathbb{R}^+}$ be a continuous stochastic process such that $\mathbb{E}[\exp(s(X_{t'} - X_t)) \mid \mathcal{F}_t] = \exp((\mathcal{V}_{t'} - \mathcal{V}_t)\varphi(s))$ almost surely for arbitrary $t' \geq t \geq 0$ and $s \in (-a, b)$, where a and b are positive numbers or infinity, and $\varphi(s)$ is a non-negative function of $s \in (-a, b)$. Let $\tau > 0$, $\gamma > 0$ and $\eta > 0$. Then, $\Pr\{\inf_{t>0}[X_t - X_0 + \gamma\mathcal{V}_t + \frac{\varphi(-s)}{s}(\mathcal{V}_t - \mathcal{V}_\tau)] \leq 0\} = [\exp(\varphi(-s) - \gamma s)]^{\mathcal{V}_\tau}$ for any $s \in (0, a)$ and $\Pr\{\sup_{t>0}[X_t - X_0 - \gamma\mathcal{V}_t - \frac{\varphi(s)}{s}(\mathcal{V}_t - \mathcal{V}_\tau)] \geq 0\} = [\exp(\varphi(s) - \gamma s)]^{\mathcal{V}_\tau}$ for any $s \in (0, b)$. In particular, under the above assumptions on X_t , \mathcal{V}_t and $\varphi(s)$, the following statements hold true:

- (I): If there exists $s^* \in (0, a)$ such that $\varphi(-s^*) = \gamma s^*$, then $\Pr\{\inf_{t>0}(X_t - X_0 + \eta + \gamma\mathcal{V}_t) \leq 0\} = e^{-\eta s^*}$.
- (II): If there exists $s^* \in (0, b)$ such that $\varphi(s^*) = \gamma s^*$, then $\Pr\{\sup_{t>0}(X_t - X_0 - \eta - \gamma\mathcal{V}_t) \geq 0\} = e^{-\eta s^*}$.
- (III): If $\varphi(s)$ is a continuous function smaller than $\gamma|s|$ at a neighborhood of 0, then $\Pr\{\inf_{t>0}[X_t - X_0 + \gamma\mathcal{V}_t + \alpha(\gamma)(\mathcal{V}_t - \mathcal{V}_\tau)] \leq 0\} = \inf_{s \in (0,a)} [\exp(\varphi(-s) - \gamma s)]^{\mathcal{V}_\tau}$ and $\Pr\{\sup_{t>0}[X_t - X_0 - \gamma\mathcal{V}_t - \beta(\gamma)(\mathcal{V}_t - \mathcal{V}_\tau)] \geq 0\} = \inf_{s \in (0,b)} [\exp(\varphi(s) - \gamma s)]^{\mathcal{V}_\tau}$.

$0\} = \inf_{s \in (0,b)} [\exp(\varphi(s) - \gamma s)]^{\nu_\tau}$, where $\alpha(\gamma)$ and $\beta(\gamma)$ are functions of γ defined as follows: $\alpha(\gamma)$ is equal to $\frac{\varphi(-s^*)}{s^*}$ if $\inf_{s \in (0,a)} [\varphi(-s) - \gamma s]$ is attained at $s^* \in (0,a)$ and otherwise equal to $\lim_{s \uparrow a} \frac{\varphi(-s)}{s}$; $\beta(\gamma)$ is equal to $\frac{\varphi(s^*)}{s^*}$ if $\inf_{s \in (0,b)} [\varphi(s) - \gamma s]$ is attained at $s^* \in (0,b)$ and otherwise equal to $\lim_{s \uparrow b} \frac{\varphi(s)}{s}$. Moreover, $0 < \alpha(\gamma) < \gamma$ and $0 < \beta(\gamma) < \gamma$.

Applying Theorem 6 to i.i.d random variables with common probability density (or mass) function in an exponential family, we have shown the following results, which generalize Chernoff bounds [4].

Corollary 5 Let Y_1, Y_2, \dots be i.i.d. random samples of Y which possesses a probability density (or mass) function $f_Y(y; \theta) = w(y) \exp(u(\theta)y - v(\theta))$ such that $\frac{dv(\theta)}{d\theta} = \theta \frac{du(\theta)}{d\theta}$ for $\theta \in \Theta$. Define $X_n = \sum_{i=1}^n Y_i$ for $n \in \mathbb{N}$. Define $\mathcal{M}(z, \theta) = \frac{\exp(u(\theta)z - v(\theta))}{\exp(u(z)z - v(z))}$ and $\rho(z, \theta, m, n) = mz + (n - m) \frac{v(z) - v(\theta)}{u(z) - u(\theta)}$ for $z, \theta \in \Theta$ and $m, n \in \mathbb{N}$. Then, for all integer $m > 0$ and real number $\gamma > 0$,

$$\Pr \left\{ \sup_{n > 0} [X_n - n\theta - \gamma(n \vee m)] \geq 0 \right\} \leq \Pr \left\{ \sup_{n > 0} [X_n - \rho(\theta + \gamma, \theta, m, n)] \geq 0 \right\} \leq [\mathcal{M}(\theta + \gamma, \theta)]^m \quad (15)$$

$$\Pr \left\{ \inf_{n > 0} [X_n - n\theta + \gamma(n \vee m)] \leq 0 \right\} \leq \Pr \left\{ \inf_{n > 0} [X_n - \rho(\theta - \gamma, \theta, m, n)] \leq 0 \right\} \leq [\mathcal{M}(\theta - \gamma, \theta)]^m, \quad (16)$$

provided that $\theta + \gamma \in \Theta$ and $\theta - \gamma \in \Theta$ respectively.

See Appendix I for a proof.

By virtue of Theorem 6, we can generalize Hoeffding-Azuma's inequality [1, 9] as follows.

Corollary 6 Let $(X_t)_{t \in \mathbb{R}^+}$ be a right-continuous stochastic process. Assume that there exist a right-continuous function \mathcal{V}_t and a stochastic process $(Y_t)_{t \in \mathbb{R}^+}$ such that for all $t \geq 0$, Y_t is measurable in \mathcal{F}_t , and that $|X_t - Y_\tau|^2 \leq \mathcal{V}_t - \mathcal{V}_\tau$ almost surely for arbitrary $t \geq \tau \geq 0$. For $\gamma > 0$ and $\tau > 0$ such that $\mathcal{V}_\tau > 0$, the following statements hold true:

(I) If $(X_t, \mathcal{F}_t)_{t \in \mathbb{R}^+}$ is a super-martingale, then

$$\Pr \left\{ \sup_{t > 0} \left[X_t - X_0 - \frac{\gamma}{2} \left(1 + \frac{\mathcal{V}_t}{\mathcal{V}_\tau} \right) \right] \geq 0 \right\} \leq \exp \left(-\frac{\gamma^2}{2\mathcal{V}_\tau} \right). \quad (17)$$

(II) If $(X_t, \mathcal{F}_t)_{t \in \mathbb{R}^+}$ is a sub-martingale, then

$$\Pr \left\{ \inf_{t > 0} \left[X_t - X_0 + \frac{\gamma}{2} \left(1 + \frac{\mathcal{V}_t}{\mathcal{V}_\tau} \right) \right] \leq 0 \right\} \leq \exp \left(-\frac{\gamma^2}{2\mathcal{V}_\tau} \right). \quad (18)$$

(III) If $(X_t, \mathcal{F}_t)_{t \in \mathbb{R}^+}$ is a martingale, then

$$\Pr \left\{ \sup_{t > 0} \left[|X_t - X_0| - \frac{\gamma}{2} \left(1 + \frac{\mathcal{V}_t}{\mathcal{V}_\tau} \right) \right] \geq 0 \right\} \leq 2 \exp \left(-\frac{\gamma^2}{2\mathcal{V}_\tau} \right). \quad (19)$$

See Appendix J for a proof.

With the help of Theorem 6, we have generalized Bernstein's inequality [3], Bennett's inequality [2] and Chernoff bound [5] as follows.

Corollary 7 Let $(X_k, \mathcal{F}_k)_{k \in \mathbb{Z}^+}$ be a martingale satisfying $\text{Var}(X_n | \mathcal{F}_{n-1}) \stackrel{\text{def}}{=} \mathbb{E}[(X_n - \mathbb{E}[X_n | \mathcal{F}_{n-1}])^2 | \mathcal{F}_{n-1}] \leq \sigma_n^2$ and $X_n - X_{n-1} \leq a_n + b$ almost surely for $n \in \mathbb{N}$, where $b > 0$ and a_n are deterministic numbers. Define $\mathcal{V}_n = \sum_{i=1}^n (\sigma_i^2 + a_i^2)$ for $n \in \mathbb{N}$. Then,

$$\Pr \left\{ \sup_{n > 0} \left[X_n - X_0 - \gamma \mathcal{V}_m - \left(\frac{\gamma}{\ln(1+b\gamma)} - \frac{1}{b} \right) (\mathcal{V}_n - \mathcal{V}_m) \right] \geq 0 \right\} \leq \left[\exp \left(\frac{\gamma}{b} - \frac{(1+b\gamma) \ln(1+b\gamma)}{b^2} \right) \right]^{\mathcal{V}_m}, \quad (20)$$

$$\Pr \left\{ \sup_{n > 0} \left[X_n - X_0 - \frac{\gamma}{2} \left(1 + \frac{\mathcal{V}_n}{\mathcal{V}_m} \right) \right] \geq 0 \right\} \leq \exp \left(-\frac{\gamma^2}{2(\mathcal{V}_m + b\gamma/3)} \right) \quad (21)$$

for all integer $m > 0$ and real number $\gamma > 0$. Specially, if $b = 1$ and $a_n = 0$ for $n \in \mathbb{N}$, then

$$\Pr \left\{ \sup_{n>0} \left[X_n - X_0 - \frac{\gamma}{2} \left(1 + \frac{\mathcal{V}_n}{\mathcal{V}_m} \right) \right] \geq 0 \right\} \leq \exp \left(-\frac{\gamma^2}{4\mathcal{V}_m} \right) \quad (22)$$

for all integer $m > 0$ and real number $\gamma \in (0, \frac{7}{2}\mathcal{V}_m)$.

See Appendix K for a proof.

Applying Theorem 6 to a Poisson process, we have obtained the following results.

Corollary 8 Let X_t be the number of arrivals in time interval $[0, t]$ for a Poisson process with an arrival rate $\lambda > 0$. Then,

$$\begin{aligned} \Pr \left\{ \sup_{t>0} \left[X_t - (\lambda + \gamma)\tau - \frac{\gamma(t - \tau)}{\ln(1 + \frac{\gamma}{\lambda})} \right] \geq 0 \right\} &\leq \left[\left(\frac{\lambda}{\lambda + \gamma} \right)^{\lambda + \gamma} e^{\gamma} \right]^{\tau}, \\ \Pr \left\{ \inf_{t>0} \left[X_t - (\lambda - \gamma)\tau + \frac{\gamma(t - \tau)}{\ln(1 - \frac{\gamma}{\lambda})} \right] \leq 0 \right\} &\leq \left[\left(\frac{\lambda}{\lambda - \gamma} \right)^{\lambda - \gamma} e^{-\gamma} \right]^{\tau} \end{aligned}$$

for any $\tau > 0$ and $\gamma > 0$.

See Appendix L for a proof.

4 Conclusion

In this paper, we have developed some new optional stopping theorems on martingale processes which require no assumption of uniform integrability and integrable stopping times. Making use of bounds of moment generating functions of increments of stochastic processes, we have established a wide class of maximal inequalities on stochastic processes, which includes classical results such as Chernoff bounds, Hoeffding-Azuma inequalities as special cases.

A Proof of Theorem 1

Throughout the proof of the theorem, let $A \in \mathcal{F}_{\tau_1}$ and $B = A \cap \{\tau_1 = n\} \in \mathcal{F}_n$.

A.1 Proof of Discrete-Time Optional Stopping Theorem under Assumption (1)

In this section of Appendix A, we shall show the discrete-time optional stopping theorem under assumption (1). More formally, we want to prove the following result:

Let $(X_k, \mathcal{F}_k)_{k \in \mathbb{Z}^+}$ be a super-martingale. Let τ_1 and τ_2 be two stopping times such that $\tau_1 \leq \tau_2$ almost surely and that there exists a constant C so that $\{\tau_2 > k\} \subseteq \{|X_k| < C\}$ for all $k \in \mathbb{Z}^+$. Assume that X_{τ_2} exist and $\mathbb{E}[|X_{\tau_2}|]$ is finite. Then, $\mathbb{E}[X_{\tau_2} | \mathcal{F}_{\tau_1}] \leq X_{\tau_1}$ and $\mathbb{E}[X_{\tau_2}] \leq \mathbb{E}[X_{\tau_1}]$ almost surely, with equality if $(X_k, \mathcal{F}_k)_{k \in \mathbb{Z}^+}$ is a martingale.

The following result stated as Lemma 1 is due to Doob [8], which can be found in many text books of probability theory.

Lemma 1 For $i \geq n$,

$$\int_{B \cap \{\tau_2 \geq n\}} X_n d\mathbb{P} \geq \int_{B \cap \{n \leq \tau_2 \leq i\}} X_{\tau_2} d\mathbb{P} + \int_{B \cap \{\tau_2 > i\}} X_i d\mathbb{P}, \quad (23)$$

with equality if X_t is a martingale.

Lemma 2 $\Pr \{ \lim_{i \rightarrow \infty} \mathbb{I}_{B \cap \{n \leq \tau_2 \leq i\}} = \mathbb{I}_{B \cap \{n \leq \tau_2 < \infty\}} \} = 1.$

Proof. To show this, consider two cases. In the case that $\omega \notin B \cap \{n \leq \tau_2 < \infty\}$, we have $\omega \notin B \cap \{n \leq \tau_2 \leq i\}$ for all $i \geq n$ and thus $\mathbb{I}_{B \cap \{n \leq \tau_2 < \infty\}} = 0 = \lim_{i \rightarrow \infty} \mathbb{I}_{B \cap \{n \leq \tau_2 \leq i\}}$. In the case that $\omega \in B \cap \{n \leq \tau_2 < \infty\}$, we have $\mathbb{I}_{B \cap \{n \leq \tau_2 < \infty\}} = 1$. Then, $\mathbb{I}_{B \cap \{n \leq \tau_2 \leq i\}} = 1$ for $i \geq \tau_2(\omega)$, which implies that $\lim_{i \rightarrow \infty} \mathbb{I}_{B \cap \{n \leq \tau_2 \leq i\}} = 1$. This proves the lemma. \square

Lemma 3 Assume that $\mathbb{E}[|X_{\tau_2}|] < \infty$. Then,

$$\lim_{i \rightarrow \infty} \int_{B \cap \{n \leq \tau_2 \leq i\}} X_{\tau_2} d\mathbb{P} = \int_{B \cap \{n \leq \tau_2 < \infty\}} X_{\tau_2} d\mathbb{P} \leq \int_{B \cap \{n \leq \tau_2 < \infty\}} |X_{\tau_2}| d\mathbb{P} < \infty.$$

Proof. From the assumption that $\mathbb{E}[|X_{\tau_2}|] < \infty$, we have

$$\mathbb{E}[X_{\tau_2} \mathbb{I}_{B \cap \{n \leq \tau_2 < \infty\}}] \leq \mathbb{E}[|X_{\tau_2}| \mathbb{I}_{B \cap \{n \leq \tau_2 < \infty\}}] \leq \mathbb{E}[|X_{\tau_2}|] < \infty.$$

By virtue of Lemma 2, we have that $X_{\tau_2} \mathbb{I}_{B \cap \{n \leq \tau_2 \leq i\}} \rightarrow X_{\tau_2} \mathbb{I}_{B \cap \{n \leq \tau_2 < \infty\}}$ almost surely as $i \rightarrow \infty$. Since $|X_{\tau_2} \mathbb{I}_{B \cap \{n \leq \tau_2 \leq i\}}| \leq |X_{\tau_2}| \mathbb{I}_{B \cap \{n \leq \tau_2 < \infty\}}$, the lemma follows from the dominated convergence theorem. \square

Lemma 4

$$\lim_{i \rightarrow \infty} \int_{B \cap \{\tau_2 > i\}} X_i d\mathbb{P} = \int_{B \cap \{\tau_2 = \infty\}} X_{\tau_2} d\mathbb{P} \leq \int_{B \cap \{\tau_2 = \infty\}} |X_{\tau_2}| d\mathbb{P} < \infty.$$

Proof. First, we shall show that

$$\lim_{i \rightarrow \infty} \int_{B \cap \{i < \tau_2 < \infty\}} X_i d\mathbb{P} = 0. \quad (24)$$

By the assumption that there exists a constant C so that $\{\tau_2 > t\} \subseteq \{|X_t| < C\}$ for any $t \geq 0$, we have

$$\begin{aligned} \lim_{i \rightarrow \infty} \left| \int_{B \cap \{i < \tau_2 < \infty\}} X_i d\mathbb{P} \right| &\leq \lim_{i \rightarrow \infty} \int_{B \cap \{i < \tau_2 < \infty\}} |X_i| d\mathbb{P} \\ &\leq \lim_{i \rightarrow \infty} \int_{B \cap \{i < \tau_2 < \infty\}} C d\mathbb{P} \leq \lim_{i \rightarrow \infty} \int_{\{i < \tau_2 < \infty\}} C d\mathbb{P} \\ &= C \lim_{i \rightarrow \infty} \Pr\{i < \tau_2 < \infty\} = 0, \end{aligned}$$

which implies (24).

Next, we shall show that

$$\lim_{i \rightarrow \infty} \int_{B \cap \{\tau_2 = \infty\}} X_i d\mathbb{P} = \int_{B \cap \{\tau_2 = \infty\}} X_{\tau_2} d\mathbb{P}. \quad (25)$$

By the assumption that there exists a constant C so that $\{\tau_2 > t\} \subseteq \{|X_t| < C\}$ for any $t \geq 0$, we have $|X_i \mathbb{I}_{B \cap \{\tau_2 = \infty\}}| \leq C$ for all $i \geq n$. By the definition of X_{τ_2} , we have that $X_i \mathbb{I}_{B \cap \{\tau_2 = \infty\}} \rightarrow X_{\tau_2} \mathbb{I}_{B \cap \{\tau_2 = \infty\}}$ as $i \rightarrow \infty$. Therefore, applying the bounded convergence theorem leads to (25) and the inequality $\int_{B \cap \{\tau_2 = \infty\}} |X_{\tau_2}| d\mathbb{P} < \infty$.

Finally, combining (24) and (25) gives

$$\lim_{i \rightarrow \infty} \int_{B \cap \{\tau_2 > i\}} X_i d\mathbb{P} = \lim_{i \rightarrow \infty} \int_{B \cap \{i < \tau_2 < \infty\}} X_i d\mathbb{P} + \lim_{i \rightarrow \infty} \int_{B \cap \{\tau_2 = \infty\}} X_i d\mathbb{P} = \int_{B \cap \{\tau_2 = \infty\}} X_{\tau_2} d\mathbb{P}.$$

This completes the proof of the lemma. \square

Now we are in a position to prove the discrete-time optional stopping theorem under assumption (1). Since X_{τ_2} exists, it must be true that X_{τ_1} exists. It suffices to show that

$$\int_{A \cap \{\tau_2 \geq \tau_1\}} X_{\tau_2} d\mathbb{P} \leq \int_{A \cap \{\tau_2 \geq \tau_1\}} X_{\tau_1} d\mathbb{P}$$

for any $A \in \mathcal{F}_{\tau_1}$. For this, in turn, it is sufficient to show that, for every $n \in \mathbb{Z}^+ \cup \{\infty\}$,

$$\int_{A \cap \{\tau_2 \geq \tau_1\} \cap \{\tau_1 = n\}} X_{\tau_2} d\mathbb{P} \leq \int_{A \cap \{\tau_2 \geq \tau_1\} \cap \{\tau_1 = n\}} X_{\tau_1} d\mathbb{P}.$$

This inequality is clearly true for $n = \infty$, because

$$A \cap \{\tau_2 \geq \tau_1\} \cap \{\tau_1 = n\} = A \cap \{\tau_1 = \infty, \tau_2 = \infty\} = A \cap \{\tau_1 = \infty, \tau_2 = \infty, X_{\tau_2} = X_{\tau_1}\}$$

holds for $n = \infty$. Recall Lemma 1, we have that for $i \geq n$,

$$\int_{B \cap \{\tau_2 \geq n\}} X_n d\mathbb{P} \geq \int_{B \cap \{n \leq \tau_2 \leq i\}} X_{\tau_2} d\mathbb{P} + \int_{B \cap \{\tau_2 > i\}} X_i d\mathbb{P}, \quad (26)$$

with equality if X_i is a martingale. Taking limits on the right side of (26) and making use of Lemmas 3 and 4, we have

$$\begin{aligned} \int_{B \cap \{\tau_2 \geq n\}} X_n d\mathbb{P} &\geq \lim_{i \rightarrow \infty} \int_{B \cap \{n \leq \tau_2 \leq i\}} X_{\tau_2} d\mathbb{P} + \lim_{i \rightarrow \infty} \int_{B \cap \{\tau_2 > i\}} X_i d\mathbb{P} \\ &= \int_{B \cap \{n \leq \tau_2 < \infty\}} X_{\tau_2} d\mathbb{P} + \int_{B \cap \{\tau_2 = \infty\}} X_{\tau_2} d\mathbb{P} = \int_{B \cap \{\tau_2 \geq n\}} X_{\tau_2} d\mathbb{P}, \end{aligned}$$

with equality if $(X_k, \mathcal{F}_k)_{k \in \mathbb{Z}^+}$ is a martingale. This completes the proof of the discrete-time optional stopping theorem under assumption (1).

A.2 Proof of Discrete-Time Optional Stopping Theorem for Super-martingale with Bounded Increment

In this section of Appendix A, we shall show the discrete-time optional stopping theorem for super-martingale with bounded increment. More formally, we want to prove the following result:

Let $(X_k, \mathcal{F}_k)_{k \in \mathbb{Z}^+}$ be a super-martingale such that there exists a constant Δ so that $|X_{k+1} - X_k| < \Delta$ almost surely for all $k \in \mathbb{Z}^+$. Let τ_1 and τ_2 be two stopping times such that $\tau_1 \leq \tau_2$ almost surely and that there exists a constant C so that $\{\tau_2 > k\} \subseteq \{|X_k| < C\}$ for all $k \in \mathbb{Z}^+$. Then, $\mathbb{E}[X_{\tau_2} | \mathcal{F}_{\tau_1}] \leq X_{\tau_1}$ and $\mathbb{E}[X_{\tau_2}] \leq \mathbb{E}[X_{\tau_1}]$ almost surely, with equality if $(X_k, \mathcal{F}_k)_{k \in \mathbb{Z}^+}$ is a martingale.

For this purpose, it suffices to show that assumption (1) is satisfied. This can be accomplished by proving the following lemma.

Lemma 5 *$X_{\tau \wedge k}$ is a UI super-martingale. Moreover, X_{τ} exists and $\mathbb{E}[|X_{\tau}|] < \infty$.*

Proof. For simplicity of notations, we denote $\int_F X d\mathbb{P}$ by $\mathbb{E}[X; F]$. Let $\Upsilon > 0$. Note that

$$\begin{aligned}
\mathbb{E}[|X_{\tau \wedge k}|; |X_{\tau \wedge k}| \geq \Upsilon] &= \mathbb{E}[|X_{\tau \wedge k}|; |X_{\tau \wedge k}| \geq \Upsilon, \tau > k] + \sum_{i=0}^k \mathbb{E}[|X_{\tau \wedge k}|; |X_{\tau \wedge k}| \geq \Upsilon, \tau = i] \\
&= \mathbb{E}[|X_k|; |X_k| \geq \Upsilon, \tau > k] + \sum_{i=0}^k \mathbb{E}[|X_i|; |X_\tau| \geq \Upsilon, \tau = i] \\
&= \mathbb{E}[|X_k|; |X_k| \geq \Upsilon, \tau > k] + \mathbb{E}[|X_0|; |X_\tau| \geq \Upsilon, \tau = 0] + \sum_{i=1}^k \mathbb{E}[|X_i|; |X_\tau| \geq \Upsilon, i-1 < \tau = i] \\
&= \mathbb{E}[|X_k|; |X_k| \geq \Upsilon, \tau > k] + \mathbb{E}[|X_0|; |X_0| \geq \Upsilon, \tau = 0] + \sum_{i=1}^k \mathbb{E}[|X_i|; |X_i| \geq \Upsilon, i-1 < \tau = i] \\
&\leq \mathbb{E}[|X_k|; \Upsilon \leq |X_k| < C] + \mathbb{E}[|X_0|; |X_0| \geq \Upsilon] + \sum_{i=1}^k \mathbb{E}[|X_i|; |X_i| \geq \Upsilon, |X_{i-1}| < C] \\
&\leq \mathbb{E}[|X_k|; \Upsilon \leq |X_k| < C] + \mathbb{E}[|X_0|; |X_0| \geq \Upsilon] + \sum_{i=1}^k \mathbb{E}[|X_i|; |X_{i-1}| + |X_i - X_{i-1}| \geq \Upsilon, |X_{i-1}| < C]
\end{aligned}$$

for all $k \geq 0$. By the bounded increment assumption, there exists a positive constant $\Delta > 0$ such that $\Pr\{|X_i - X_{i-1}| < \Delta\} = 1$ for $i \geq 1$. Choose $\Upsilon > C + \Delta$, then $\mathbb{E}[|X_{\tau \wedge k}|; |X_{\tau \wedge k}| \geq \Upsilon] \leq \mathbb{E}[|X_0|; |X_0| \geq \Upsilon]$ for all $k \geq 0$. By the assumption that $\mathbb{E}[|X_0|] < \infty$, we have that there exists a sufficiently large $\Upsilon > C + \Delta$ such that $\mathbb{E}[|X_0|; |X_0| \geq \Upsilon] < \varepsilon$. Hence, $\mathbb{E}[|X_{\tau \wedge k}|; |X_{\tau \wedge k}| \geq \Upsilon] < \varepsilon$ for all $k \geq 0$. This implies that $X_{\tau \wedge k}$ is a UI super-martingale.

Since $X_{\tau \wedge k}$ is a UI super-martingale, we have $\sup_k \mathbb{E}[|X_{\tau \wedge k}|] < \infty$. Moreover, $X_{\tau \wedge k}$ converges as $k \rightarrow \infty$ and X_τ exists almost surely. By Fatou's lemma, $\mathbb{E}[|X_\tau|] = \mathbb{E}[\liminf_{k \rightarrow \infty} |X_{\tau \wedge k}|] \leq \liminf_{k \rightarrow \infty} \mathbb{E}[|X_{\tau \wedge k}|] \leq \sup_k \mathbb{E}[|X_{\tau \wedge k}|] < \infty$. □

A.3 Proof of Discrete-Time Optional Stopping Theorem for Non-negative Super-martingale

In this section of Appendix A, we shall show the discrete-time optional stopping theorem for non-negative super-martingale. More formally, we want to prove the following result:

Let $(X_k, \mathcal{F}_k)_{k \in \mathbb{Z}^+}$ be a non-negative super-martingale. Let τ_1 and τ_2 be two stopping times such that $\tau_1 \leq \tau_2$ almost surely and that there exists a constant C so that $\{\tau_2 > k\} \subseteq \{X_k < C\}$ for all $k \in \mathbb{Z}^+$. Then, $\mathbb{E}[X_{\tau_2} | \mathcal{F}_{\tau_1}] \leq X_{\tau_1}$ and $\mathbb{E}[X_{\tau_2}] \leq \mathbb{E}[X_{\tau_1}]$ almost surely, with equality if $(X_k, \mathcal{F}_k)_{k \in \mathbb{Z}^+}$ is a martingale.

Since a non-negative super-martingale must converge, it follows that both X_{τ_1} and X_{τ_2} exist. As an immediate consequence of Lemma 4, we have the following result.

Lemma 6 *For any non-negative integer n , $0 \leq \lim_{i \rightarrow \infty} \int_{B \cap \{\tau_2 > i\}} X_i d\mathbb{P} = \int_{B \cap \{\tau_2 = \infty\}} X_{\tau_2} d\mathbb{P} < \infty$.*

We are now in a position to prove the discrete-time optional stopping theorem for non-negative super-martingale. It suffices to show that

$$\int_{A \cap \{\tau_2 \geq \tau_1\}} X_{\tau_2} d\mathbb{P} \leq \int_{A \cap \{\tau_2 \geq \tau_1\}} X_{\tau_1} d\mathbb{P}$$

for any $A \in \mathcal{F}_{\tau_1}$. For this, in turn, it is sufficient to show that, for every $n \in \mathbb{Z}^+ \cup \{\infty\}$,

$$\int_{A \cap \{\tau_2 \geq \tau_1\} \cap \{\tau_1 = n\}} X_{\tau_2} d\mathbb{P} \leq \int_{A \cap \{\tau_2 \geq \tau_1\} \cap \{\tau_1 = n\}} X_{\tau_1} d\mathbb{P}.$$

This inequality is clearly true for $n = \infty$, because

$$A \cap \{\tau_2 \geq \tau_1\} \cap \{\tau_1 = n\} = A \cap \{\tau_1 = \infty, \tau_2 = \infty\} = A \cap \{\tau_1 = \infty, \tau_2 = \infty, X_{\tau_2} = X_{\tau_1}\}$$

holds for $n = \infty$. It remains to show, for $n \in \mathbb{Z}^+$,

$$\int_{B \cap \{\tau_2 \geq n\}} X_{\tau_2} d\mathbb{P} \leq \int_{B \cap \{\tau_2 \geq n\}} X_n d\mathbb{P}. \quad (27)$$

As a consequence of Lemma 2,

$$\int_{B \cap \{n \leq \tau_2 < \infty\}} X_{\tau_2} d\mathbb{P} = \mathbb{E} \left[\liminf_{i \rightarrow \infty} X_{\tau_2} \mathbb{I}_{B \cap \{n \leq \tau_2 \leq i\}} \right]. \quad (28)$$

Note that

$$\mathbb{E} \left[\liminf_{i \rightarrow \infty} X_{\tau_2} \mathbb{I}_{B \cap \{n \leq \tau_2 \leq i\}} \right] \leq \liminf_{i \rightarrow \infty} \mathbb{E} [X_{\tau_2} \mathbb{I}_{B \cap \{n \leq \tau_2 \leq i\}}] \quad (29)$$

$$\begin{aligned} &= \liminf_{i \rightarrow \infty} \int_{B \cap \{n \leq \tau_2 \leq i\}} X_{\tau_2} d\mathbb{P} \\ &\leq \liminf_{i \rightarrow \infty} \left[\int_{B \cap \{\tau_2 \geq n\}} X_n d\mathbb{P} - \int_{B \cap \{\tau_2 > i\}} X_i d\mathbb{P} \right] \end{aligned} \quad (30)$$

$$\begin{aligned} &= \int_{B \cap \{\tau_2 \geq n\}} X_n d\mathbb{P} - \limsup_{i \rightarrow \infty} \int_{B \cap \{\tau_2 > i\}} X_i d\mathbb{P} \\ &= \int_{B \cap \{\tau_2 \geq n\}} X_n d\mathbb{P} - \int_{B \cap \{\tau_2 = \infty\}} X_{\tau_2} d\mathbb{P}, \end{aligned} \quad (31)$$

where (29) follows from Fatou's lemma, (30) follows from Lemma 1, and (31) follows from Lemma 6. By the assumption that $(X_k, \mathcal{F}_k)_{k \in \mathbb{Z}^+}$ is a super-martingale, we have that X_n is integrable and thus $0 \leq \int_{B \cap \{\tau_2 \geq n\}} X_n d\mathbb{P} < \infty$. From Lemma 6, we know that $0 \leq \int_{B \cap \{\tau_2 = \infty\}} X_{\tau_2} d\mathbb{P} < \infty$. It follows from (31) that $0 \leq \mathbb{E} [\liminf_{i \rightarrow \infty} X_{\tau_2} \mathbb{I}_{B \cap \{n \leq \tau_2 \leq i\}}] < \infty$. Combining (28) and (31) yields

$$0 \leq \int_{B \cap \{n \leq \tau_2 < \infty\}} X_{\tau_2} d\mathbb{P} \leq \int_{B \cap \{\tau_2 \geq n\}} X_n d\mathbb{P} - \int_{B \cap \{\tau_2 = \infty\}} X_{\tau_2} d\mathbb{P} < \infty$$

or equivalently,

$$0 \leq \int_{B \cap \{\tau_2 \geq n\}} X_{\tau_2} d\mathbb{P} - \int_{B \cap \{\tau_2 = \infty\}} X_{\tau_2} d\mathbb{P} \leq \int_{B \cap \{\tau_2 \geq n\}} X_n d\mathbb{P} - \int_{B \cap \{\tau_2 = \infty\}} X_{\tau_2} d\mathbb{P} < \infty,$$

which implies (27). So, we have established that $\mathbb{E}[X_{\tau_2} | \mathcal{F}_{\tau_1}] \leq X_{\tau_1}$ and $\mathbb{E}[X_{\tau_2}] \leq \mathbb{E}[X_{\tau_1}]$ for the case that $(X_k, \mathcal{F}_k)_{k \in \mathbb{Z}^+}$ is a super-martingale. It follows that $0 \leq \mathbb{E}[X_{\tau_2}] \leq \mathbb{E}[X_0] < \infty$. Applying Theorem 1, we have $\mathbb{E}[X_{\tau_2} | \mathcal{F}_{\tau_1}] = X_{\tau_1}$ and $\mathbb{E}[X_{\tau_2}] = \mathbb{E}[X_{\tau_1}]$ for the case that $(X_k, \mathcal{F}_k)_{k \in \mathbb{Z}^+}$ is a martingale. Thus, we have established the discrete-time optional stopping theorem for non-negative super-martingale.

So, we have completed the proof of Theorem 1.

B Proof of Theorem 2

Recall the assumption that there exist constants δ and Δ such that $|X_{t'} - X_t| < \Delta$ almost surely provided that $|t' - t| < \delta$. For such δ , define $\nu = \lceil \log_2 \frac{1}{\delta} \rceil$ and $\mathbb{T}_n = \{k2^{-(n+\nu)} : k \in \mathbb{Z}^+\}$ for $n \in \mathbb{N}$. Define

$$\begin{aligned} \rho_n &= \inf\{t \in \mathbb{T}_n : X_t \notin \mathcal{D}_t\}, & \varrho_n &= \inf\{t \in \mathbb{T}_n : X_t \notin \mathcal{D}_t\}, \\ X_{\rho_n} &= \lim_{\substack{t \rightarrow \infty \\ t \in \mathbb{T}_n}} X_{\rho_n \wedge t}, & X_{\varrho_n} &= \lim_{\substack{t \rightarrow \infty \\ t \in \mathbb{T}_n}} X_{\varrho_n \wedge t}, \\ S_n &= 2^{n+\nu} \rho_n, & T_n &= 2^{n+\nu} \varrho_n \end{aligned}$$

for $n \in \mathbb{N}$. Define

$$t_{k,n} = k2^{-(n+\nu)}, \quad Y_k^n = X_{t_{k,n}}, \quad \mathcal{F}_k^n = \mathcal{F}_{t_{k,n}}$$

for $k \in \mathbb{Z}^+$ and $n \in \mathbb{N}$. Then, the following statements are true.

- (a): For $n \in \mathbb{N}$, $(Y_k^n, \mathcal{F}_k^n)_{k \in \mathbb{Z}^+}$ is a discrete-time super-martingale.
- (b): For $k \in \mathbb{Z}^+$ and $n \in \mathbb{N}$, the inequality $t_{k+1,n} - t_{k,n} < \delta$ holds and consequently, $|Y_{k+1}^n - Y_k^n| < \Delta$ almost surely.
- (c): $S_n \leq T_n$ and $\{T_n > k\} \subseteq \{|Y_k^n| > C\}$ for $n \in \mathbb{N}$.

Define

$$Y_{S_n}^n = \lim_{k \rightarrow \infty} Y_{S_n \wedge k}^n, \quad Y_{T_n}^n = \lim_{k \rightarrow \infty} Y_{T_n \wedge k}^n$$

for $n \in \mathbb{N}$. Note that S_n and T_n are stopping times non-increasing with respect to $n \in \mathbb{N}$. To complete the proof of the theorem, we need the following results.

Lemma 7 (I) ρ_n and ϱ_n are stopping times non-increasing with respect to $n \in \mathbb{N}$.

- (II) $\rho_n \geq \tau_1$, $\varrho_n \geq \tau_2$ and $\rho_n \leq \varrho_n$ for all $n \in \mathbb{N}$.
- (III) $\lim_{n \rightarrow \infty} \rho_n = \tau_1$ and $\lim_{n \rightarrow \infty} \varrho_n = \tau_2$.
- (IV) For all $n \in \mathbb{N}$, X_{ρ_n} and X_{ϱ_n} exist almost surely.
- (V) X_{τ_1} and X_{τ_2} exist almost surely.
- (VI) As n tends to infinity, X_{ρ_n} and X_{ϱ_n} converge to X_{τ_1} and X_{τ_2} respectively and almost surely.

Proof. Statements (I) – (III) are obviously true. We shall show statements (IV), (V) and (VI).

Proof of Statement (IV) : Consider the existence of X_{ρ_n} for $n \in \mathbb{N}$. From Lemma 5 and statements (a), (b) and (c) appeared before Lemma 7, we know that for every $n \in \mathbb{N}$, $(Y_{S_n \wedge k}^n, \mathcal{F}_k^n)_{k \in \mathbb{Z}^+}$ is a discrete-time UI martingale and it follows that, almost surely, $\lim_{k \rightarrow \infty} Y_{S_n \wedge k}^n$ exists and is finite. By the definitions of \mathbb{T}_n , ρ_n , S_n and $\{Y_k^n\}$, we have that $\lim_{\substack{t \rightarrow \infty \\ t \in \mathbb{T}_n}} X_{\rho_n \wedge t} = \lim_{k \rightarrow \infty} Y_{S_n \wedge k}^n$ almost surely for all $n \in \mathbb{N}$. Since X_{ρ_n} is defined as $\lim_{\substack{t \rightarrow \infty \\ t \in \mathbb{T}_n}} X_{\rho_n \wedge t}$, it follows that X_{ρ_n} exists almost surely for all $n \in \mathbb{N}$. In a similar manner, the existence of X_{ϱ_n} can be established for $n \in \mathbb{N}$.

Proof of Statement (V) : Consider the existence of X_{τ_1} . Let $n \in \mathbb{N}$ be fixed and let $\omega \in \Omega$ with $\tau_1(\omega) = \infty$. From the proof of Statement (IV), we know that the limit $\lim_{\substack{t \rightarrow \infty \\ t \in \mathbb{T}_n}} X_{\rho_n \wedge t}$ exists. Since $\rho_n \geq \tau_1$, we have $\rho_n(\omega) = \infty$. This implies that $\lim_{\substack{t \rightarrow \infty \\ t \in \mathbb{T}_n}} X_t(\omega)$ exists. We claim that the limit $\lim_{t \rightarrow \infty} X_t(\omega)$ exists and is equal to $\lim_{\substack{t \rightarrow \infty \\ t \in \mathbb{T}_n}} X_t(\omega)$. Suppose, to get a contradiction, that $\lim_{t \rightarrow \infty} X_t(\omega)$ does not exist. Then, there exist an $\varepsilon > 0$ and a sequence $\{t_i\}_{i=1}^\infty$ with $t_1 < t_2 < t_3 < \dots$ and $t_i \rightarrow \infty$ as $i \rightarrow \infty$, such that $|X_{t_i} - c| > \varepsilon$ for $i \geq 1$, where c denotes $\lim_{\substack{t \rightarrow \infty \\ t \in \mathbb{T}_n}} X_t(\omega)$. Define $\mathbb{W}_n = \mathbb{T}_n \cup \{t_i\}_{i=1}^\infty$. That is, the sequence $\{t_i\}_{i=1}^\infty$ is added to \mathbb{T}_n to form a new sequence \mathbb{W}_n . Define $\mu_n = \inf\{t \in \mathbb{W}_n : X_t \notin \mathcal{D}_t\}$. By the same argument as that for proving the existence of $\lim_{\substack{t \rightarrow \infty \\ t \in \mathbb{T}_n}} X_{\rho_n \wedge t}$, we can show

that $\lim_{\substack{t \rightarrow \infty \\ t \in \mathbb{W}_n}} X_{\mu_n \wedge t}$ exists almost surely. Observing that $\mu_n \geq \tau_1$, we have $\mu_n(\omega) = \infty$. Therefore, $\lim_{\substack{t \rightarrow \infty \\ t \in \mathbb{W}_n}} X_t(\omega)$ exists. Since $\mathbb{T}_n \subseteq \mathbb{W}_n$, it must be true that $\lim_{\substack{t \rightarrow \infty \\ t \in \mathbb{W}_n}} X_t(\omega) = c$. Since $\{t_i\}_{i=1}^\infty \subseteq \mathbb{W}_n$, it follows that $\lim_{i \rightarrow \infty} X_{t_i}(\omega)$ exists and is equal to c . This contradicts to the assumption that $|X_{t_i} - c| > \varepsilon$ for $i \geq 1$. Thus, we have established the claim that $\lim_{t \rightarrow \infty} X_t(\omega)$ exists and is equal to $\lim_{\substack{t \rightarrow \infty \\ t \in \mathbb{T}_n}} X_t(\omega)$ for $\omega \in \Omega$ with $\tau_1(\omega) = \infty$. Since X_{τ_1} is defined as $\lim_{t \rightarrow \infty} X_{\tau_1 \wedge t}$, it follows that X_{τ_1} exists almost surely. In a similar manner, the existence of X_{τ_2} can be established.

Proof of Statement (IV) : Consider the convergence of $(X_{\rho_n})_{n \in \mathbb{N}}$. Recall the established fact that $X_{\rho_n} = \lim_{\substack{t \rightarrow \infty \\ t \in \mathbb{T}_n}} X_{\rho_n \wedge t}$ exists almost surely for all $n \in \mathbb{N}$. Let $\omega \in \Omega$ with $\tau_1(\omega) = \infty$. Since $\rho_n \geq \tau_1$, we have $\rho_n(\omega) = \infty$ for all $n \in \mathbb{N}$. It follows that $X_{\rho_n}(\omega) = \lim_{\substack{t \rightarrow \infty \\ t \in \mathbb{T}_n}} X_t(\omega) = \lim_{t \rightarrow \infty} X_t(\omega)$ for all $n \in \mathbb{N}$. Therefore,

$$\lim_{n \rightarrow \infty} X_{\rho_n}(\omega) = \lim_{t \rightarrow \infty} X_t(\omega) \quad \text{for } \omega \in \Omega \text{ with } \tau_1(\omega) = \infty. \quad (32)$$

Now let $\omega \in \Omega$ with $\tau_1(\omega) < \infty$. Since $\lim_{n \rightarrow \infty} \rho_n = \tau_1$, we have that $\rho_n(\omega) < \infty$ for sufficiently large $n \in \mathbb{N}$. This implies that $X_{\rho_n}(\omega) = X_{\rho_n(\omega)}(\omega)$ for sufficiently large $n \in \mathbb{N}$. Since $\lim_{n \rightarrow \infty} \rho_n(\omega) = \tau_1(\omega)$ and $(X_t)_{t \in \mathbb{R}^+}$ is a right-continuous process, we have that

$$\lim_{n \rightarrow \infty} X_{\rho_n}(\omega) = X_{\tau_1(\omega)}(\omega) \quad \text{for } \omega \in \Omega \text{ with } \tau_1(\omega) < \infty. \quad (33)$$

Making use of (32), (33) and the definition of X_{τ_1} , we have that X_{ρ_n} converges to X_{τ_1} almost surely. Similarly, we can show that X_{ρ_n} converges to X_{τ_2} almost surely.

□

Lemma 8 *As n tends to infinity, X_{ρ_n} and X_{ρ_n} converge to X_{τ_1} and X_{τ_2} respectively in L^1 .*

Proof. Let $\{\mathcal{G}_{-n} : n \in \mathbb{N} \cup \{\infty\}\}$ be a collection of sub- σ -algebras of \mathcal{F} with $\mathcal{G}_{-n} = \mathcal{F}_{\rho_n}$ for $n \in \mathbb{N}$ and $\mathcal{G}_{-\infty} = \bigcap_{k \in \mathbb{N}} \mathcal{G}_{-k}$. Define $Z_{-n} = X_{\rho_n}$ for $n \in \mathbb{N}$. Then, $Z_{-n} = Y_{S_n}^n$ for $n \in \mathbb{N}$. According to Theorem 1, we have that $\mathbb{E}[Y_{S_n}^n | \mathcal{F}_{S_{n+1}}^{n+1}] \leq Y_{S_{n+1}}^{n+1}$ almost surely for $n \in \mathbb{N}$. Since $Y_{S_n}^n = X_{\rho_n}$ and $\mathcal{F}_{S_n}^n = \mathcal{F}_{\rho_n}$ for $n \in \mathbb{N}$, we have that $\mathbb{E}[X_{\rho_n} | \mathcal{F}_{\rho_{n+1}}] \leq X_{\rho_{n+1}}$ almost surely for $n \in \mathbb{N}$. This implies that $\mathbb{E}[Z_{-n} | \mathcal{G}_{-(n+1)}] \leq Z_{-(n+1)}$ almost surely. Hence, $\{Z_{-n}, n \in \mathbb{N}\}$ is a supermartingale relative to $\{\mathcal{G}_{-n} : n \in \mathbb{N} \cup \{\infty\}\}$ in the context of Lévy-Doob Downward Theorem (see, e.g., [10, page 148–149]). Moreover, from Theorem 1, we have that $\mathbb{E}[Y_{S_n}^n] \leq \mathbb{E}[Y_{0,n}]$ almost surely for $n \in \mathbb{N}$. Since $Y_{S_n}^n = X_{\rho_n}$ and $Y_{0,n} = X_0$, we have $\mathbb{E}[X_{\rho_n}] \leq \mathbb{E}[X_0] < \infty$, which implies that $\sup_{n \in \mathbb{N}} \mathbb{E}[Z_{-n}] = \sup_{n \in \mathbb{N}} \mathbb{E}[X_{\rho_n}] < \infty$. Therefore, it follows from Lévy-Doob Downward Theorem that $\{Z_{-n}, n \in \mathbb{N}\}$ is uniformly integrable and that the limit $Z_{-\infty} \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} Z_{-n}$ exists almost surely and $\mathbb{E}[|Z_{-\infty}|] < \infty$. From Lemma 7, we know that $Z_{-\infty} = X_{\tau_1}$. Since $\{Z_{-n}, n \in \mathbb{N}\}$ is uniformly integrable and converges to X_{τ_1} almost surely, it follows that $X_{\rho_n} = Z_{-n} \rightarrow X_{\tau_1}$ in L^1 . Similarly, we have that $X_{\rho_n} \rightarrow X_{\tau_2}$ in L^1 .

□

We are now in a position to prove the theorem. We follow the classical argument for extending the optional stopping theorem from discrete UI martingale to continuous-time UI martingale. According to Theorem 1, we have $\mathbb{E}[Y_{T_n}^n | \mathcal{F}_{S_n}^n] \leq Y_{S_n}^n$ almost surely for $n \in \mathbb{N}$. Since $Y_{S_n}^n = X_{\rho_n}$, $Y_{T_n}^n = X_{\rho_n}$ and $\mathcal{F}_{S_n}^n = \mathcal{F}_{\rho_n}$, we have

$$\mathbb{E}[X_{\rho_n} | \mathcal{F}_{\rho_n}] \leq X_{\rho_n} \quad (34)$$

almost surely for $n \in \mathbb{N}$. Since $\tau_1 \leq \rho_n$, it holds that $\mathcal{F}_{\tau_1} \subseteq \mathcal{F}_{\rho_n}$. It follows from (34) and the tower property that

$$\mathbb{E}[X_{\rho_n} \mid \mathcal{F}_{\tau_1}] = \mathbb{E}[\mathbb{E}[X_{\rho_n} \mid \mathcal{F}_{\rho_n}] \mid \mathcal{F}_{\tau_1}] \leq \mathbb{E}[X_{\rho_n} \mid \mathcal{F}_{\tau_1}] \quad (35)$$

almost surely. Let $\mathcal{E} \in \mathcal{F}_{\tau_1}$. It follows from (35) that $\int_{\mathcal{E}} X_{\rho_n} d\mathbb{P} \leq \int_{\mathcal{E}} X_{\rho_n} d\mathbb{P}$. Invoking Lemma 8, we have $X_{\rho_n} \rightarrow X_{\tau_1}$ in L^1 and consequently,

$$\left| \int_{\mathcal{E}} X_{\rho_n} d\mathbb{P} - \int_{\mathcal{E}} X_{\tau_1} d\mathbb{P} \right| \leq \int_{\mathcal{E}} |X_{\rho_n} - X_{\tau_1}| d\mathbb{P} \leq \mathbb{E}[|X_{\rho_n} - X_{\tau_1}|] \rightarrow 0$$

as $n \rightarrow \infty$. This implies that $\int_{\mathcal{E}} X_{\rho_n} d\mathbb{P} \rightarrow \int_{\mathcal{E}} X_{\tau_1} d\mathbb{P}$ as $n \rightarrow \infty$. Similarly, $\int_{\mathcal{E}} X_{\rho_n} d\mathbb{P} \rightarrow \int_{\mathcal{E}} X_{\tau_2} d\mathbb{P}$ as $n \rightarrow \infty$. Therefore, $\int_{\mathcal{E}} X_{\tau_2} d\mathbb{P} = \lim_{n \rightarrow \infty} \int_{\mathcal{E}} X_{\rho_n} d\mathbb{P} \leq \lim_{n \rightarrow \infty} \int_{\mathcal{E}} X_{\rho_n} d\mathbb{P} = \int_{\mathcal{E}} X_{\tau_1} d\mathbb{P}$. Since the argument holds for arbitrary $\mathcal{E} \in \mathcal{F}_{\tau_1}$, the proof of the theorem is thus completed.

C Proof of Theorem 3

Since $(X_t, \mathcal{F}_t)_{t \in \mathbb{R}^+}$ is a nonnegative, right-continuous supermartingale, the limit $\lim_{t \rightarrow \infty} X_t$ exists almost surely. As a consequence of this fact and the definition that $X_{\tau_i} = \lim_{t \rightarrow \infty} X_{\tau_i \wedge t}$, $i = 1, 2$, it must be true that both X_{τ_1} and X_{τ_2} exist almost surely. For $n \in \mathbb{N}$, define

$$\begin{aligned} \rho_n &= \inf\{t \in \mathbb{S}_n : X_t \notin \mathcal{D}_t\}, & \varrho_n &= \inf\{t \in \mathbb{S}_n : X_t \notin \mathcal{D}_t\}, \\ X_{\rho_n} &= \lim_{\substack{t \rightarrow \infty \\ t \in \mathbb{S}_n}} X_{\rho_n \wedge t}, & X_{\varrho_n} &= \lim_{\substack{t \rightarrow \infty \\ t \in \mathbb{S}_n}} X_{\varrho_n \wedge t}, \\ S_n &= 2^{n+\nu} \rho_n, & T_n &= 2^{n+\nu} \varrho_n. \end{aligned}$$

By the same argument as that for the existence of X_{τ_1} and X_{τ_2} , we have that X_{ρ_n} and X_{ϱ_n} exist almost surely for all $n \in \mathbb{N}$. Define

$$t_{k,n} = k2^{-n}, \quad Y_k^n = X_{t_{k,n}}, \quad \mathcal{F}_k^n = \mathcal{F}_{t_{k,n}}$$

for $k \in \mathbb{Z}^+$ and $n \in \mathbb{N}$. Then, the following statements are true:

- (a): For $n \in \mathbb{N}$, $(Y_k^n, \mathcal{F}_k^n)_{k \in \mathbb{Z}^+}$ is a discrete-time non-negative super-martingale.
- (b): $S_n \leq T_n$ and $\{T_n > k\} \subseteq \{|Y_k^n| > C\}$ for $n \in \mathbb{N}$.

Define

$$Y_{S_n}^n = \lim_{k \rightarrow \infty} Y_{S_n \wedge k}^n, \quad Y_{T_n}^n = \lim_{k \rightarrow \infty} Y_{T_n \wedge k}^n$$

for $n \in \mathbb{N}$. Note that S_n and T_n are stopping times non-increasing with respect to $n \in \mathbb{N}$. To complete the proof of the theorem, we need to use the following results.

Lemma 9 (I) ρ_n and ϱ_n are stopping times non-increasing with respect to $n \in \mathbb{N}$.

(II) $\rho_n \geq \tau_1$, $\varrho_n \geq \tau_2$ and $\rho_n \leq \varrho_n$ for all $n \in \mathbb{N}$.

(III) $\lim_{n \rightarrow \infty} \rho_n = \tau_1$ and $\lim_{n \rightarrow \infty} \varrho_n = \tau_2$.

(IV) As n tends to infinity, X_{ρ_n} and X_{ϱ_n} converge to X_{τ_1} and X_{τ_2} respectively and almost surely.

Proof. Statements (I) – (III) are obvious from the definition. We shall show statement (IV). Consider the convergence of $(X_{\rho_n})_{n \in \mathbb{N}}$. Since $(X_t, \mathcal{F}_t)_{t \in \mathbb{R}^+}$ is a non-negative supermartingale, the limit $\lim_{t \rightarrow \infty} X_t(\omega)$ must exist for $\omega \in \Omega$. Let $\omega \in \Omega$ with $\tau_1(\omega) = \infty$. Since $\rho_n \geq \tau_1$, we have $\rho_n(\omega) = \infty$ for all $n \in \mathbb{N}$. It follows that $X_{\rho_n}(\omega) = \lim_{\substack{t \rightarrow \infty \\ t \in \mathbb{S}_n}} X_t(\omega) = \lim_{t \rightarrow \infty} X_t(\omega)$ for all $n \in \mathbb{N}$. Therefore,

$$\lim_{n \rightarrow \infty} X_{\rho_n}(\omega) = \lim_{t \rightarrow \infty} X_t(\omega) \quad \text{for } \omega \in \Omega \text{ with } \tau_1(\omega) = \infty. \quad (36)$$

Now let $\omega \in \Omega$ with $\tau_1(\omega) < \infty$. Since $\lim_{n \rightarrow \infty} \rho_n = \tau_1$, we have that $\rho_n(\omega) < \infty$ for sufficiently large $n \in \mathbb{N}$. This implies that $X_{\rho_n}(\omega) = X_{\rho_n(\omega)}(\omega)$ for sufficiently large $n \in \mathbb{N}$. Since $\lim_{n \rightarrow \infty} \rho_n(\omega) = \tau_1(\omega)$ and $(X_t)_{t \in \mathbb{R}^+}$ is a right-continuous process, we have that

$$\lim_{n \rightarrow \infty} X_{\rho_n}(\omega) = X_{\tau_1(\omega)}(\omega) \quad \text{for } \omega \in \Omega \text{ with } \tau_1(\omega) < \infty. \quad (37)$$

Making use of (36), (37) and the definition of X_{τ_1} , we have that X_{ρ_n} converges to X_{τ_1} almost surely. Similarly, we can show that X_{ρ_n} converges to X_{τ_2} almost surely. \square

Making use of Theorem 1, Lemma 9 and a similar technique as that for proving Lemma 8, we can show the following result.

Lemma 10 *As n tends to infinity, X_{ρ_n} and X_{ρ_n} converge to X_{τ_1} and X_{τ_2} respectively in L^1 .*

Finally, the proof of Theorem 3 can be completed by making use of Theorem 1, Lemma 10 and a similar technique as that for proving Theorem 2.

D Proof of Theorem 4

First, let $\gamma > c$ and consider $\Pr \{ \sup_{t \geq 0} X_t \geq \gamma \}$. Define $\tau = \inf \{ t \in [0, \infty) : X_t \geq \gamma \}$. Then, τ is a stopping time. Define X_τ such that for $\omega \in \Omega$,

$$X_\tau(\omega) = \begin{cases} X_{\tau(\omega)}(\omega) & \text{if } \tau(\omega) < \infty, \\ c & \text{if } \tau(\omega) = \infty \end{cases}$$

Since $(X_t, \mathcal{F}_t)_{t \in \mathbb{R}^+}$ is a right-continuous UI martingale which converges to c , we have $\mathbb{E}[X_\tau] < \infty$ and $\mathbb{E}[X_\tau] = \mathbb{E}[X_0] = c$. It follows that

$$\gamma \Pr \{ \tau < \infty \} = \int_{\{\tau < \infty\}} \gamma d\mathbb{P} \leq \int_{\{\tau < \infty\}} X_\tau d\mathbb{P} = \mathbb{E}[X_\tau] - \mathbb{E}[X_\tau \mathbb{I}_{\{\tau = \infty\}}] = c - \mathbb{E}[c \mathbb{I}_{\{\tau = \infty\}}] = c \Pr \{ \tau < \infty \},$$

which implies that $\Pr \{ \tau < \infty \} = 0$. Since $X_t \rightarrow c < \gamma$ almost surely, we have $\Pr \{ \limsup_{t \geq 0} X_t \geq \gamma \} = 0$, which implies that $\Pr \{ \sup_{t \geq 0} X_t \geq \gamma \} = \Pr \{ \tau < \infty \}$. Therefore,

$$\Pr \left\{ \sup_{t \geq 0} X_t \geq \gamma \right\} = 0 \quad \text{for } \gamma > c. \quad (38)$$

Now let $\gamma < c$ and consider $\Pr \{ \inf_{t \geq 0} X_t \leq \gamma \}$. Making use of (38) and the observation that $(-X_t, \mathcal{F}_t)_{t \in \mathbb{R}^+}$ is a right-continuous UI martingale which converges to $-c$, we have that $\Pr \{ \sup_{t \geq 0} (-X_t) \geq (-\gamma) \} = 0$ for $-\gamma > -c$, which implies that

$$\Pr \left\{ \inf_{t \geq 0} X_t \leq \gamma \right\} = 0 \quad \text{for } \gamma < c. \quad (39)$$

Combining (38) and (39) gives $\Pr \{ X_t = c \text{ for all } t \in [0, \infty) \} = 1$. This completes the proof of the theorem.

E Proof of Theorem 5

Let $\gamma > c$ and consider $\Pr \{ \sup_{t \geq 0} X_t \geq \gamma \}$. Define stopping time $\tau = \inf \{ t \in [0, \infty) : X_t \geq \gamma \}$. By definition of X_τ ,

$$X_\tau(\omega) = \begin{cases} X_{\tau(\omega)}(\omega) & \text{if } \tau(\omega) < \infty \\ c & \text{if } \tau(\omega) = \infty \end{cases}$$

for $\omega \in \Omega$. For simplicity of notations, let $\mu = \mathbb{E}[X_0]$. Since $X_t \rightarrow c < \gamma$ almost surely, we have $\Pr\{\limsup_{t \geq 0} X_t \geq \gamma\} = 0$ and thus $\Pr\{\sup_{t \geq 0} X_t \geq \gamma\} = \Pr\{\tau < \infty\}$. Since $(X_t, \mathcal{F}_t)_{t \in \mathbb{R}^+}$ is a right-continuous, non-negative super-martingale, it follows from Theorem 3 that $\mathbb{E}[X_\tau] \leq \mathbb{E}[X_0] = \mu$. Therefore,

$$\begin{aligned} \gamma \Pr\{\tau < \infty\} &= \int_{\{\tau < \infty\}} \gamma d\mathbb{P} \leq \int_{\{\tau < \infty\}} X_\tau d\mathbb{P} = \mathbb{E}[X_\tau] - \mathbb{E}[X_\tau \mathbb{I}_{\{\tau = \infty\}}] \\ &\leq \mu - \mathbb{E}[X_\tau \mathbb{I}_{\{\tau = \infty\}}] = \mu - \mathbb{E}[c \mathbb{I}_{\{\tau = \infty\}}] = \mu - c(1 - \Pr\{\tau < \infty\}). \end{aligned}$$

So, we have established the inequality $\gamma \Pr\{\tau < \infty\} \leq \mu - c(1 - \Pr\{\tau < \infty\})$. Since $\gamma > c$, solving this inequality with respect to $\Pr\{\tau < \infty\}$ yields $\Pr\{\tau < \infty\} \leq \frac{\mu - c}{\gamma - c}$. It follows that $\Pr\{\sup_{t \geq 0} X_t \geq \gamma\} \leq \frac{\mu - c}{\gamma - c}$ for $\gamma > c$, which implies that $c \leq \mu$.

Now consider $\Pr\{\sup_{t \geq 0} X_t \geq \gamma\}$ under additional assumption that $(X_t, \mathcal{F}_t)_{t \in \mathbb{R}^+}$ is a continuous martingale. Since $(X_t, \mathcal{F}_t)_{t \in \mathbb{R}^+}$ is a continuous martingale, we have $\gamma \Pr\{\tau < \infty\} = \int_{\{\tau < \infty\}} X_\tau d\mathbb{P} = \mathbb{E}[X_\tau] - \mathbb{E}[X_\tau \mathbb{I}_{\{\tau = \infty\}}] = \mu - \mathbb{E}[X_\tau \mathbb{I}_{\{\tau = \infty\}}] = \mu - c(1 - \Pr\{\tau < \infty\})$ for $\gamma > c$. Consequently, $\Pr\{\sup_{t \geq 0} X_t \geq \gamma\} = \Pr\{\tau < \infty\} = \frac{\mu - c}{\gamma - c}$ for $\gamma > c$. Recalling that $c \leq \mu = \mathbb{E}[X_0]$, we have that $\Pr\{\sup_{t \geq 0} X_t \geq \gamma\} = \frac{\mathbb{E}[X_0] - c}{\gamma - c}$ for $\gamma > \mathbb{E}[X_0]$. It remains to show that $\Pr\{\sup_{t \geq 0} X_t \geq \gamma\} = 1$ for $\gamma \leq \mathbb{E}[X_0]$.

We claim that $\Pr\{\sup_{t \geq 0} X_t \geq \mu\} = 1$. In the case of $c = \mu$, if $\Pr\{\sup_{t \geq 0} X_t \geq \mu\} < 1$, then there exists $\epsilon > 0$ such that $\Pr\{X_t < \mu - \epsilon \text{ for all } t \geq 0\} > 0$, which contradicts to the fact that $X_t \rightarrow \mu = c$ almost surely. In the cases of $c < \mu$, by the established result, $\Pr\{\sup_{t \geq 0} X_t \geq \mu\} = \frac{\mu - c}{\mu - c} = 1$. This proves the claim. Consequently, we have $1 \geq \Pr\{\sup_{t \geq 0} X_t \geq \gamma\} \geq \Pr\{\sup_{t \geq 0} X_t \geq \mu\} = 1$ for $\gamma \leq \mu = \mathbb{E}[X_0]$. This completes the proof of the theorem.

F Proof of Corollary 4

By the assumption of the theorem, it can be readily shown that $(Y_t, \mathcal{F}_t)_{t \in \mathbb{R}^+}$ is a non-negative martingale. It follows that $(Y_t)_{t \in \mathbb{R}^+}$ converges almost surely. We claim that for $s \in (0, b)$, $\lim_{t \rightarrow \infty} Y_t = 0$ almost surely. To show this claim, it suffices to show that for $s \in (0, b)$, $\lim_{n \rightarrow \infty} Y_n = 0$ almost surely, where the limit is taken under the constraint that $n \in \mathbb{N}$. Let $\gamma > 0$ and $s \in (0, b)$. For $n \in \mathbb{N}$ and $\theta \in (0, s)$, we have $\mathbb{E}[\exp(\theta X_n)] \leq \exp(\varphi(\theta)\mathcal{V}_n)$ and by Markov inequality,

$$\begin{aligned} \Pr\{Y_n \geq \gamma\} &= \Pr\left\{\exp(\theta X_n) \geq \exp\left(\frac{\theta}{s}[\ln \gamma + \varphi(s)\mathcal{V}_n]\right)\right\} \\ &\leq \frac{\exp(\varphi(\theta)\mathcal{V}_n)}{\exp\left(\frac{\theta}{s}[\ln \gamma + \varphi(s)\mathcal{V}_n]\right)} = \left(\frac{1}{\gamma}\right)^{\theta/s} \exp\left(\left[\frac{\varphi(\theta)}{\theta} - \frac{\varphi(s)}{s}\right]\theta\mathcal{V}_n\right). \end{aligned}$$

By the assumption that $\liminf_{n \rightarrow \infty} (\mathcal{V}_{n+1} - \mathcal{V}_n) > 0$, we have that $\mathcal{V}_n > 0$ for large enough $n \in \mathbb{N}$. Since $\varphi(s)\mathcal{V}_n$ is a convex function of s , it follows that $\varphi(s)$ is a convex function, which implies that $\frac{\varphi(\theta)}{\theta} - \frac{\varphi(s)}{s} < 0$. Since $\liminf_{n \rightarrow \infty} (\mathcal{V}_{n+1} - \mathcal{V}_n) > 0$, there exists a real number $d > 0$ and an $m \in \mathbb{N}$ such that $\mathcal{V}_{n+1} - \mathcal{V}_n > d$ for all $n \geq m$. Thus, $\mathcal{V}_n > \mathcal{V}_m + (n - m)d$ for $n > m$. It follows that

$$\begin{aligned} \sum_{n \in \mathbb{N}} \exp\left(\left[\frac{\varphi(\theta)}{\theta} - \frac{\varphi(s)}{s}\right]\theta\mathcal{V}_n\right) &< \sum_{n=1}^{m-1} \exp\left(\left[\frac{\varphi(\theta)}{\theta} - \frac{\varphi(s)}{s}\right]\theta\mathcal{V}_n\right) + \sum_{n=m}^{\infty} \exp\left(\left[\frac{\varphi(\theta)}{\theta} - \frac{\varphi(s)}{s}\right]\theta[\mathcal{V}_m + (n - m)d]\right) \\ &= \sum_{n=1}^{m-1} \exp\left(\left[\frac{\varphi(\theta)}{\theta} - \frac{\varphi(s)}{s}\right]\theta\mathcal{V}_n\right) + \frac{\exp\left(\left[\frac{\varphi(\theta)}{\theta} - \frac{\varphi(s)}{s}\right]\theta\mathcal{V}_m\right)}{1 - \exp\left(\left[\frac{\varphi(\theta)}{\theta} - \frac{\varphi(s)}{s}\right]\theta d\right)} < \infty. \end{aligned}$$

This implies that $\sum_{n \in \mathbb{N}} \Pr\{Y_n \geq \gamma\}$ is finite. It follows from Borel-Cantelli lemma that $\Pr\{\cap_{n=1}^{\infty} \cup_{k \geq n} [Y_k \geq \gamma]\} = 0$ and thus $\lim_{n \rightarrow \infty} Y_n = 0$ almost surely for $s \in (0, b)$. This proves the claim that for $s \in (0, b)$, $\lim_{t \rightarrow \infty} Y_t = 0$ almost surely. Similarly, we can show that for $s \in (-a, 0)$, $\lim_{t \rightarrow \infty} Y_t = 0$ almost surely.

Since for $s \in (-a, 0) \cup (0, b)$, $(Y_t, \mathcal{F}_t)_{t \in \mathbb{R}^+}$ is a non-negative continuous martingale which converges to 0, the proof of the theorem can be completed by applying Theorem 5.

G Proof of Theorem 6

Define $W_t = \exp(s(X_t - X_0) - \varphi(s)\mathcal{V}_t)$ for $t \geq 0$ and $s \in (-a, b)$. Then, for all $s \in (-a, b)$ and arbitrary $t' \geq t \geq 0$, we have

$$\begin{aligned}\mathbb{E}[W_{t'} \mid \mathcal{F}_t] &= \mathbb{E}[\exp(s(X_{t'} - X_0) - \varphi(s)\mathcal{V}_{t'}) \mid \mathcal{F}_t] = \mathbb{E}[\exp(s(X_{t'} - X_t) - \varphi(s)(\mathcal{V}_{t'} - \mathcal{V}_t)) W_t \mid \mathcal{F}_t] \\ &= W_t \exp(-\varphi(s)(\mathcal{V}_{t'} - \mathcal{V}_t)) \mathbb{E}[\exp(s(X_{t'} - X_t)) \mid \mathcal{F}_t] \leq W_t.\end{aligned}$$

Hence, for any $s \in (-a, b)$, $(W_t, \mathcal{F}_t)_{t \in \mathbb{R}^+}$ is a super-martingale with $\mathbb{E}[W_0] = \mathbb{E}[\exp(-\varphi(s)\mathcal{V}_0)] \leq 1$. By the assumption on the continuity of the sample paths of $\{sX_t - \varphi(s)\mathcal{V}_t\}_{t \geq 0}$, we have that almost all sample paths of $(W_t)_{t \in \mathbb{R}^+}$ is right-continuous.

To prove (3), note that for any $s \in (0, a)$ and real number $\gamma > 0$,

$$\begin{aligned}& \Pr \left\{ \inf_{t \geq 0} \left[X_t - X_0 + \gamma \mathcal{V}_t + \frac{\varphi(-s)}{s}(\mathcal{V}_t - \mathcal{V}_\tau) \right] \leq 0 \right\} \\ &= \Pr \left\{ \inf_{t \geq 0} \left[X_t - X_0 + \gamma \mathcal{V}_t + \frac{\varphi(-s)}{s}(\mathcal{V}_t - \mathcal{V}_\tau) \right] s \leq 0 \right\} \\ &= \Pr \left\{ \inf_{t \geq 0} [s(X_t - X_0) + \varphi(-s)\mathcal{V}_t + \gamma s \mathcal{V}_t - \varphi(-s)\mathcal{V}_\tau] \leq 0 \right\} \\ &= \Pr \left\{ \sup_{t \geq 0} [-s(X_t - X_0) - \varphi(-s)\mathcal{V}_t - \gamma s \mathcal{V}_t + \varphi(-s)\mathcal{V}_\tau] \geq 0 \right\} \\ &= \Pr \left\{ \sup_{t \geq 0} [-s(X_t - X_0) - \varphi(-s)\mathcal{V}_t] \geq \gamma s \mathcal{V}_\tau - \varphi(-s)\mathcal{V}_\tau \right\} \\ &= \Pr \left\{ \sup_{t \geq 0} W_t \geq \exp(\gamma s \mathcal{V}_\tau - \varphi(-s)\mathcal{V}_\tau) \right\} \tag{40}\end{aligned}$$

$$\begin{aligned}& \leq \exp(\varphi(-s)\mathcal{V}_\tau - \gamma s \mathcal{V}_\tau) \tag{41} \\ &= [\exp(\varphi(-s) - \gamma s)]^{\mathcal{V}_\tau}.\end{aligned}$$

Here, we have used the definition of W_t in (40). The inequality (41) follows from the super-martingale inequality. This proves (3).

To prove (4), note that for any $s \in (0, b)$ and real number $\gamma > 0$,

$$\begin{aligned}& \Pr \left\{ \sup_{t \geq 0} \left[X_t - X_0 - \gamma \mathcal{V}_t - \frac{\varphi(s)}{s}(\mathcal{V}_t - \mathcal{V}_\tau) \right] \geq 0 \right\} \\ &= \Pr \left\{ \sup_{t \geq 0} \left[X_t - X_0 - \gamma \mathcal{V}_t - \frac{\varphi(s)}{s}(\mathcal{V}_t - \mathcal{V}_\tau) \right] s \geq 0 \right\} \\ &= \Pr \left\{ \sup_{t \geq 0} [s(X_t - X_0) - \varphi(s)\mathcal{V}_t - \gamma s \mathcal{V}_t + \varphi(s)\mathcal{V}_\tau] \geq 0 \right\} \\ &= \Pr \left\{ \sup_{t \geq 0} [s(X_t - X_0) - \varphi(s)\mathcal{V}_t] \geq \gamma s \mathcal{V}_\tau - \varphi(s)\mathcal{V}_\tau \right\} \\ &= \Pr \left\{ \sup_{t \geq 0} W_t \geq \exp(\gamma s \mathcal{V}_\tau - \varphi(s)\mathcal{V}_\tau) \right\} \tag{42}\end{aligned}$$

$$\begin{aligned}& \leq \exp(\varphi(s)\mathcal{V}_\tau - \gamma s \mathcal{V}_\tau) \tag{43} \\ &= [\exp(\varphi(s) - \gamma s)]^{\mathcal{V}_\tau}.\end{aligned}$$

Here, we have used the definition of W_t in (42). The inequality (43) follows from the super-martingale inequality. This proves (4).

Before proving (5) and (6), we shall show (9) and (10). Note that, as a consequence of $0 \leq \varphi(s) \leq \gamma s$, we have $\varphi(s)(\mathcal{V}_t - \mathcal{V}_\tau) \leq \gamma s(\mathcal{V}_\tau \vee \mathcal{V}_t - \mathcal{V}_\tau)$ or equivalently, $\varphi(s)\mathcal{V}_t + \gamma s\mathcal{V}_\tau - \varphi(s)\mathcal{V}_\tau \leq \gamma s(\mathcal{V}_\tau \vee \mathcal{V}_t)$ for any $t > 0$. This inequality can be written as

$$\varphi(s)\mathcal{V}_t + s(\eta + \gamma\mathcal{V}_\tau) - \varphi(s)\mathcal{V}_\tau \leq \eta s + \gamma s(\mathcal{V}_\tau \vee \mathcal{V}_t). \quad (44)$$

Hence, for any $s \in (0, b)$,

$$\begin{aligned} & \Pr \left\{ \sup_{t>0} [X_t - X_0 - \eta - \gamma(\mathcal{V}_\tau \vee \mathcal{V}_t)] \geq 0 \right\} \\ &= \Pr \left\{ \sup_{t>0} [s(X_t - X_0) - \eta s - \gamma s(\mathcal{V}_\tau \vee \mathcal{V}_t)] \geq 0 \right\} \\ &\leq \Pr \left\{ \sup_{t>0} [s(X_t - X_0) - \varphi(s)\mathcal{V}_t - s(\eta + \gamma\mathcal{V}_\tau) + \varphi(s)\mathcal{V}_\tau] \geq 0 \right\} \end{aligned} \quad (45)$$

$$\begin{aligned} &= \Pr \left\{ \sup_{t>0} [s(X_t - X_0) - \varphi(s)\mathcal{V}_t] \geq s(\eta + \gamma\mathcal{V}_\tau) - \varphi(s)\mathcal{V}_\tau \right\} \\ &= \Pr \left\{ \sup_{t>0} W_t \geq \exp(s(\eta + \gamma\mathcal{V}_\tau) - \varphi(s)\mathcal{V}_\tau) \right\} \end{aligned} \quad (46)$$

$$\begin{aligned} &\leq \exp(\varphi(s)\mathcal{V}_\tau - s(\eta + \gamma\mathcal{V}_\tau)) \\ &= e^{-\eta s} [\exp(\varphi(s) - \gamma s)]^{\mathcal{V}_\tau}. \end{aligned} \quad (47)$$

Here, (45) follows from (44). We have used the definition of W_t in (46). Recall that, for any $s \in (-a, b)$, $(W_t, \mathcal{F}_t)_{t \in \mathbb{R}^+}$ is a super-martingale with $\mathbb{E}[W_0] \leq 1$. The inequality (47) follows from the super-martingale inequality. This proves (10). The proof of (9) is similar.

Now we are in position to prove (5) and (6). In the case that $\{s \in (0, b) : \varphi(s) \leq \gamma s\}$ is empty, (6) is clearly true, since the infimum is no less than 1. In the case that $\{s \in (0, b) : \varphi(s) \leq \gamma s\}$ is not empty, it follows from (10) that

$$\Pr \left\{ \sup_{t>0} [X_t - X_0 - \gamma(\mathcal{V}_\tau \vee \mathcal{V}_t)] \geq 0 \right\} \leq \inf_{\{s \in (0, b) : \varphi(s) \leq \gamma s\}} [\exp(\varphi(s) - \gamma s)]^{\mathcal{V}_\tau} = \inf_{s \in (0, b)} [\exp(\varphi(s) - \gamma s)]^{\mathcal{V}_\tau}.$$

This proves (6). The proof of (5) is similar.

To prove (7), note that, for any $s \in \mathcal{A}$,

$$\begin{aligned} & \Pr \left\{ \inf_{t>0} [X_t - X_0 + \eta + \gamma\mathcal{V}_t] \leq 0 \right\} = \Pr \left\{ \sup_{t>0} [(-s)(X_t - X_0) - \eta s - \gamma s\mathcal{V}_t] \geq 0 \right\} \\ &\leq \Pr \left\{ \sup_{t>0} [(-s)(X_t - X_0) - \varphi(-s)\mathcal{V}_t] \geq s\eta \right\} = \Pr \left\{ \sup_{t>0} W_t \geq \exp(s\eta) \right\} \leq \exp(-s\eta), \end{aligned}$$

where $W_t = \exp[(-s)(X_t - X_0) - \varphi(-s)\mathcal{V}_t]$ and the last inequality follows from the super-martingale inequality.

To prove (8), note that, for any $s \in \mathcal{B}$,

$$\begin{aligned} & \Pr \left\{ \sup_{t>0} [X_t - X_0 - \eta - \gamma\mathcal{V}_t] \geq 0 \right\} = \Pr \left\{ \sup_{t>0} [s(X_t - X_0) - \eta s - \gamma s\mathcal{V}_t] \geq 0 \right\} \\ &\leq \Pr \left\{ \sup_{t>0} [s(X_t - X_0) - \varphi(s)\mathcal{V}_t] \geq s\eta \right\} = \Pr \left\{ \sup_{t>0} W_t \geq \exp(s\eta) \right\} \leq \exp(-s\eta), \end{aligned}$$

where $W_t = \exp[s(X_t - X_0) - \varphi(s)\mathcal{V}_t]$ and the last inequality follows from the super-martingale inequality.

We shall show statement (I). For simplicity of notation, define $g(s) = \varphi(s) - \gamma s$ for $s \in (0, b)$. To show (12), recall the assumption that $\varphi(s)$ is a non-negative, continuous function smaller than $\gamma|s|$ at a neighborhood of 0. Hence, there exists a number $\epsilon \in (0, b)$ such that $\varphi(s) < \gamma s$ for $s \in (0, \epsilon]$. Since $g(\epsilon) < 0 = g(0)$, it must be true that either the infimum of $g(s)$ is attained at some $s^* \in (0, b)$ or $\inf_{s \in (0, b)} g(s) = \lim_{s \uparrow b} g(s) < g(\epsilon) < 0$. In the former case, (12) of statement (I) is true as a consequence of (4). In the latter case, we can define $\varphi(b) = \lim_{s \uparrow b} \varphi(s)$. Then, $b \lim_{s \uparrow b} \frac{\varphi(s)}{s} = \varphi(b) < \gamma b$ and $\varphi(b) - \gamma b = \inf_{s \in (0, b)} [\varphi(s) - \gamma s]$. Consider $W_t = \exp(b(X_t - X_0) - \varphi(b)\mathcal{V}_t)$ for $t \geq 0$. For arbitrary $t' \geq t \geq 0$, making use of Fatou's lemma, we have

$$\begin{aligned} \mathbb{E}[\exp(b(X_{t'} - X_t)) \mid \mathcal{F}_t] &= \mathbb{E}\left[\liminf_{s \uparrow b} \exp(s(X_{t'} - X_t)) \mid \mathcal{F}_t\right] \leq \liminf_{s \uparrow b} \mathbb{E}[\exp(s(X_{t'} - X_t)) \mid \mathcal{F}_t] \\ &\leq \liminf_{s \uparrow b} \exp(\varphi(s)(\mathcal{V}_{t'} - \mathcal{V}_t)) = \exp(\varphi(b)(\mathcal{V}_{t'} - \mathcal{V}_t)) \end{aligned}$$

and consequently,

$$\begin{aligned} \mathbb{E}[W_{t'} \mid \mathcal{F}_t] &= \mathbb{E}[\exp(b(X_{t'} - X_0) - \varphi(b)\mathcal{V}_{t'}) \mid \mathcal{F}_t] = \mathbb{E}[\exp(b(X_{t'} - X_t) - \varphi(b)(\mathcal{V}_{t'} - \mathcal{V}_t)) W_t \mid \mathcal{F}_t] \\ &= W_t \exp(-\varphi(b)(\mathcal{V}_{t'} - \mathcal{V}_t)) \mathbb{E}[\exp(b(X_{t'} - X_t)) \mid \mathcal{F}_t] \leq W_t \end{aligned}$$

almost surely. This implies that $(W_t, \mathcal{F}_t)_{t \in \mathbb{R}^+}$ is a super-martingale with $\mathbb{E}[W_0] = \mathbb{E}[\exp(-\varphi(b)\mathcal{V}_0)] \leq 1$. So, for any real number $\gamma > 0$,

$$\begin{aligned} &\Pr\left\{\sup_{t \geq 0} \left[X_t - X_0 - \gamma\mathcal{V}_t - \lim_{s \uparrow b} \frac{\varphi(s)}{s}(\mathcal{V}_t - \mathcal{V}_\tau)\right] \geq 0\right\} \\ &= \Pr\left\{\sup_{t \geq 0} \left[X_t - X_0 - \gamma\mathcal{V}_t - \lim_{s \uparrow b} \frac{\varphi(s)}{s}(\mathcal{V}_t - \mathcal{V}_\tau)\right] b \geq 0\right\} \\ &= \Pr\left\{\sup_{t \geq 0} [b(X_t - X_0) - \varphi(b)\mathcal{V}_t - \gamma b\mathcal{V}_\tau + \varphi(b)\mathcal{V}_\tau] \geq 0\right\} \\ &= \Pr\left\{\sup_{t \geq 0} [b(X_t - X_0) - \varphi(b)\mathcal{V}_t] \geq \gamma b\mathcal{V}_\tau - \varphi(b)\mathcal{V}_\tau\right\} \\ &= \Pr\left\{\sup_{t \geq 0} W_t \geq \exp(\gamma b\mathcal{V}_\tau - \varphi(b)\mathcal{V}_\tau)\right\} \leq [\exp(\varphi(b) - \gamma b)]^{\mathcal{V}_\tau} = \inf_{s \in (0, b)} [\exp(\varphi(s) - \gamma s)]^{\mathcal{V}_\tau}. \end{aligned}$$

This establishes (12). Now we shall show that $0 < \beta(\gamma) < \gamma$. Clearly, $\beta(\gamma)$ is positive. In the case that the infimum of $g(s)$ is attained at some $s^* \in (0, b)$, we have $g(s^*) < g(\epsilon) < 0$, which implies that $\beta(\gamma) < \gamma$. In the case that $\inf_{s \in (0, b)} g(s) = \lim_{s \uparrow b} g(s) < 0$, we have $\frac{\lim_{s \uparrow b} \varphi(s)}{b} = \lim_{s \uparrow b} \frac{\varphi(s)}{s} < \gamma$. So, in both cases, $0 < \beta(\gamma) < \gamma$.

In a similar manner, we can show (11) and the inequality $0 < \alpha(\gamma) < \gamma$.

Finally, we need to show statement (II). This is clearly true for the case that $\lim_{s \uparrow b} \frac{\varphi(s)}{s} \leq \gamma$. It suffices to consider the case that $\lim_{s \uparrow b} \frac{\varphi(s)}{s} > \gamma$. In this case, by the assumption that $\frac{\varphi(s)}{|s|}$ is monotonically increasing with respect to $|s| > 0$, there exists a unique number $b^* \in (0, b)$ such that $\frac{\varphi(b^*)}{b^*} = \gamma$. Therefore, $\inf_{\{s \in (0, b) : \varphi(s) \leq \gamma s\}} \exp(\varphi(s) - \gamma s) = \inf_{s \in (0, b^*]} \exp(\varphi(s) - \gamma s)$. It follows from (10) that (14) is true. In a similar manner, we can show (13). This concludes the proof of the theorem.

H An Upper Bound for the Moment Generating Function of a Uniform Random Variable

In this appendix, we shall establish the following result.

Theorem 8 *Let Y be a random variable uniformly distributed over $[-\frac{1}{2}, \frac{1}{2}]$. Then, $\mathbb{E}[\exp(sY)] \leq \exp\left(\frac{s^2}{24}\right)$ for any $s \in \mathbb{R}$. That is, the moment generating function of Y is bounded from above by $\exp\left(\frac{s^2}{24}\right)$.*

Proof. Note that the moment generating function of Y is $\mathbb{E}[\exp(sY)] = \frac{(e^s - 1)e^{-\frac{s}{2}}}{s}$. We want to show that $g(s) < \exp\left(\frac{s^2}{24}\right)$ for any $s \in \mathbb{R}$. Define $h(s) = s \left[\mathbb{E}[\exp(sY)] - \exp\left(\frac{s^2}{24}\right) \right]$. Then, $h(s) = (e^s - 1)e^{-\frac{s}{2}} - s \exp\left(\frac{s^2}{24}\right)$. It can be checked that the derivative of $h(s)$ is $h'(s) = u(s) - v(s)$, where

$$u(s) = \frac{1}{2} \left(e^{\frac{s}{2}} + e^{-\frac{s}{2}} \right), \quad v(s) = \left(1 + \frac{s^2}{12} \right) \exp\left(\frac{s^2}{24}\right).$$

Using Taylor series expansion formula, we can write

$$u(s) = 1 + \sum_{i=1}^{\infty} \frac{s^{2i}}{4^i (2i)!}, \quad v(s) = 1 + \sum_{i=1}^{\infty} \left[\frac{\left(\frac{s^2}{24}\right)^i}{i!} + \frac{\left(\frac{s^2}{24}\right)^{i-1} \frac{s^2}{12}}{(i-1)!} \right] = 1 + \sum_{i=1}^{\infty} \frac{1+2i}{24^i i!} s^{2i}.$$

Since $h'(0) = u(0) - v(0)$, to show the theorem, it suffices to show that $u(s) < v(s)$. This can be accomplished by proving $\frac{s^{2i}}{4^i (2i)!} < \frac{1+2i}{24^i i!} s^{2i}$ for $i = 1, 2, \dots$, or equivalently,

$$\frac{1}{(2i)!} \leq \frac{1+2i}{6^i i!}, \quad i = 1, 2, \dots$$

To do so, define the ratio of $\frac{1+2i}{6^i i!}$ to $\frac{1}{(2i)!}$ as $f(i)$. It can be checked that $f(i) = \frac{(2i)!(1+2i)}{6^i i!}$ for $i = 1, 2, \dots$. Since $f(1) \geq 1$ and $\frac{f(i+1)}{f(i)} = 1 + \frac{2i}{3} > 1$ for $k \geq 1$, we have $f(i) > 1$ for all $i > 1$. This implies that $u(s) < v(s)$ for $s \in \mathbb{R}$. The proof of the theorem is thus completed. \square

I Proof of Corollary 5

Let $\psi(\cdot)$ be the inverse function of $u(\cdot)$ such that $u(\psi(\zeta)) = \zeta$ for $\zeta \in \{u(\theta) : \theta \in \Theta\}$. Define $h(\zeta) = v(\psi(\zeta))$ for $\zeta \in \{u(\theta) : \theta \in \Theta\}$. Putting $\zeta = u(\theta)$, we have $\mathbb{E}[\exp(sY)] = \exp(h(\zeta + s) - h(\zeta))$. Define $\varphi(s) = h(\zeta + s) - h(\zeta) - \theta s$ and

$$Z_n = X_n - n\theta, \quad \mathcal{V}_n = n$$

for $n \in \mathbb{N}$. For $n \in \mathbb{N}$, let \mathcal{F}_n denote the σ -algebra generated by X_1, \dots, X_n . Then,

$$\mathbb{E}[\exp(s(Z_n - Z_m)) \mid \mathcal{F}_m] = \exp((\mathcal{V}_n - \mathcal{V}_m)\varphi(s))$$

for $m, n \in \mathbb{N}$ with $m \leq n$. Assume that $\theta + \gamma \in \Theta$. Noting that

$$\frac{dh(\zeta)}{d\zeta} = \frac{dv}{d\psi} \frac{d\psi}{d\zeta} = \psi \frac{du}{d\psi} \frac{d\psi}{d\zeta} = \psi \frac{du}{d\zeta} = \psi(\zeta), \quad (48)$$

we have

$$\frac{d[\varphi(s) - \gamma s]}{ds} = \frac{dh(\zeta + s)}{ds} - (\theta + \gamma) = \psi(\zeta + s) - (\theta + \gamma).$$

Define $s^* = u(\theta + \gamma) - u(\theta)$. Invoking the definition that $\zeta = u(\theta)$, we have $\zeta + s^* = u(\theta + \gamma)$, which implies that $\psi(\zeta + s^*) = \theta + \gamma$. Therefore,

$$\left. \frac{d[\varphi(s) - \gamma s]}{ds} \right|_{s=s^*} = \psi(\zeta + s^*) - (\theta + \gamma) = 0.$$

Since $\varphi(s) - \gamma s$ is a convex function of s , the infimum of $\varphi(s) - \gamma s$ is attained at $s = s^*$. Note that

$$\varphi(s^*) = h(\zeta + u(\theta + \gamma) - u(\theta)) - h(\zeta) - \theta[u(\theta + \gamma) - u(\theta)] = v(\theta + \gamma) - v(\theta) - \theta[u(\theta + \gamma) - u(\theta)].$$

Thus,

$$\varphi(s^*) - \gamma s^* = v(\theta + \gamma) - v(\theta) + (\theta + \gamma)[u(\theta) - u(\theta + \gamma)], \quad (49)$$

$$\frac{\varphi(s^*)}{s^*} = \frac{v(\theta + \gamma) - v(\theta)}{u(\theta + \gamma) - u(\theta)} - \theta \in (0, \gamma). \quad (50)$$

Applying Theorem 6, we have

$$\Pr \left\{ \sup_{n>0} [X_n - n\theta - \gamma(n \vee m)] \geq 0 \right\} \leq \Pr \left\{ \sup_{n>0} \left[X_n - n\theta - m\gamma - \frac{\varphi(s^*)}{s^*}(n - m) \right] \geq 0 \right\} \leq \exp(m[\varphi(s^*) - \gamma s^*]). \quad (51)$$

Making use of (49), (50), (51) and the definitions of ρ and \mathcal{M} , we have that $\Pr\{\sup_{n>0}[X_n - n\theta - \gamma(n \vee m)] \geq 0\} \leq \Pr\{\sup_{n>0}[X_n - \rho(\theta + \gamma, \theta, m, n)] \geq 0\} \leq [\mathcal{M}(\theta + \gamma, \theta)]^m$ provided that $\theta + \gamma \in \Theta$. This proves (15).

Now we shall show (16). Assume that $\theta - \gamma \in \Theta$. By virtue of (48), we have

$$\frac{d[\varphi(-s) - \gamma s]}{ds} = \frac{dh(\zeta - s)}{ds} + (\theta - \gamma) = -\psi(\zeta - s) + (\theta - \gamma).$$

Define $s^* = u(\theta) - u(\theta - \gamma)$. Invoking the definition that $\zeta = u(\theta)$, we have $\zeta - s^* = u(\theta - \gamma)$, which implies that $\psi(\zeta - s^*) = \theta - \gamma$. Therefore,

$$\left. \frac{d[\varphi(-s) - \gamma s]}{ds} \right|_{s=s^*} = \theta - \gamma - \psi(\zeta - s^*) = 0.$$

Since $\varphi(-s) - \gamma s$ is a convex function of s , the infimum of $\varphi(-s) - \gamma s$ is attained at $s = s^*$. Note that

$$\varphi(-s^*) = h(\zeta - u(\theta) + u(\theta - \gamma)) - h(\zeta) + \theta[u(\theta) - u(\theta - \gamma)] = v(\theta - \gamma) - v(\theta) + \theta[u(\theta) - u(\theta - \gamma)].$$

Thus,

$$\varphi(-s^*) - \gamma s^* = v(\theta - \gamma) - v(\theta) + (\theta - \gamma)[u(\theta) - u(\theta - \gamma)], \quad (52)$$

$$\frac{\varphi(-s^*)}{s^*} = \frac{v(\theta - \gamma) - v(\theta)}{u(\theta) - u(\theta - \gamma)} + \theta \in (0, \gamma). \quad (53)$$

Applying Theorem 6, we have

$$\Pr \left\{ \inf_{n>0} [X_n - n\theta + \gamma(n \vee m)] \leq 0 \right\} \leq \Pr \left\{ \inf_{n>0} \left[X_n - n\theta + m\gamma + \frac{\varphi(-s^*)}{s^*}(n - m) \right] \leq 0 \right\} \leq \exp(m[\varphi(-s^*) - \gamma s^*]). \quad (54)$$

Making use of (52), (53), (54) and the definitions of ρ and \mathcal{M} , we have that $\Pr\{\inf_{n>0}[X_n - n\theta + \gamma(n \vee m)] \leq 0\} \leq \Pr\{\inf_{n>0}[X_n - \rho(\theta - \gamma, \theta, m, n)] \leq 0\} \leq [\mathcal{M}(\theta - \gamma, \theta)]^m$ provided that $\theta - \gamma \in \Theta$. This proves (16). The proof of the theorem is thus completed.

J Proof of Corollary 6

For simplicity of notations, define $A_{t,\tau} = Y_\tau - X_\tau - \sqrt{\mathcal{V}_t - \mathcal{V}_\tau}$ and $B_{t,\tau} = Y_\tau - X_\tau + \sqrt{\mathcal{V}_t - \mathcal{V}_\tau} + \delta$, where δ is a positive number introduced for the purpose of ensuring $B_{t,\tau} - A_{t,\tau} > 0$. By the assumption that

$|X_t - Y_\tau|^2 \leq \mathcal{V}_t - \mathcal{V}_\tau$, we have that $A_{t,\tau} \leq X_t - X_\tau < B_{t,\tau}$ almost surely. For $t \geq \tau \geq 0$, we have

$$\mathbb{E} \left[e^{s(X_t - X_\tau)} \mid \mathcal{F}_\tau \right] \leq \mathbb{E} \left[\frac{B_{t,\tau} - (X_t - X_\tau)}{B_{t,\tau} - A_{t,\tau}} e^{sA_{t,\tau}} + \frac{(X_t - X_\tau) - A_{t,\tau}}{B_{t,\tau} - A_{t,\tau}} e^{sB_{t,\tau}} \mid \mathcal{F}_\tau \right] \quad (55)$$

$$= \frac{B_{t,\tau} e^{sA_{t,\tau}} - A_{t,\tau} e^{sB_{t,\tau}}}{B_{t,\tau} - A_{t,\tau}} + \frac{e^{sB_{t,\tau}} - e^{sA_{t,\tau}}}{B_{t,\tau} - A_{t,\tau}} \mathbb{E}[X_t - X_\tau \mid \mathcal{F}_\tau] \quad (56)$$

$$\leq \frac{B_{t,\tau} e^{sA_{t,\tau}} - A_{t,\tau} e^{sB_{t,\tau}}}{B_{t,\tau} - A_{t,\tau}} \quad (57)$$

$$\leq \exp \left(\frac{s^2 (B_{t,\tau} - A_{t,\tau})^2}{8} \right) \quad (58)$$

$$= \exp \left(\frac{s^2 (\sqrt{\mathcal{V}_t - \mathcal{V}_\tau} + \frac{\delta}{2})^2}{2} \right) \quad (59)$$

almost surely, where (55) follows from the convexity of the exponential function, (56) follows from the fact that $A_{t,\tau}$ and $B_{t,\tau}$ are measurable in \mathcal{F}_τ , (57) follows from the assumption that $(X_t, \mathcal{F}_t)_{t \in \mathbb{R}^+}$ is a super-martingale, (58) follows from the inequality $ye^x - xe^y \leq (y - x) \exp(|y - x|^2/8)$ for $y \geq x$. Since (59) holds almost surely for any $\delta > 0$, it must be true that for any $s \in \mathbb{R}$,

$$\mathbb{E} \left[e^{s(X_t - X_\tau)} \mid \mathcal{F}_\tau \right] \leq \lim_{\delta \downarrow 0} \exp \left(\frac{s^2 (\sqrt{\mathcal{V}_t - \mathcal{V}_\tau} + \frac{\delta}{2})^2}{2} \right) = \exp \left(\frac{(\mathcal{V}_t - \mathcal{V}_\tau) s^2}{2} \right)$$

almost surely. Therefore, we have shown that $\mathbb{E} [e^{s(X_t - X_\tau)} \mid \mathcal{F}_\tau] \leq \exp((\mathcal{V}_t - \mathcal{V}_\tau)\varphi(s))$ with $\varphi(s) = \frac{s^2}{2}$. Clearly, $\varphi(0) = 0$ and $\varphi(s)$ is convex. Moreover, for $\varepsilon > 0$, $\inf_{s \in (0, \infty)} [\varphi(s) - \varepsilon s]$ is attained at $s = s^*$ with $s^* = \varepsilon$. It can be checked that $\varphi(s^*) - \varepsilon s^* = -\frac{\varepsilon^2}{2}$ and $\frac{\varphi(s^*)}{s^*} = \frac{\varepsilon}{2}$. Hence, applying Theorem 6, we have

$$\Pr \left\{ \sup_{t > 0} \left[X_t - X_0 - \varepsilon \mathcal{V}_\tau - \frac{\varepsilon}{2} (\mathcal{V}_t - \mathcal{V}_\tau) \right] \geq 0 \right\} \leq \exp \left(-\frac{\varepsilon^2 \mathcal{V}_\tau}{2} \right).$$

Substituting γ in the above inequality with $\varepsilon \mathcal{V}_\tau$ yields (17). In a similar manner, we can show (18).

To show (19), note that

$$\begin{aligned} & \left\{ \sup_{t > 0} \left[X_t - X_0 - \frac{\gamma}{2} \left(1 + \frac{\mathcal{V}_t}{\mathcal{V}_\tau} \right) \right] < 0 \right\} \cap \left\{ \inf_{t > 0} \left[X_t - X_0 + \frac{\gamma}{2} \left(1 + \frac{\mathcal{V}_t}{\mathcal{V}_\tau} \right) \right] < 0 \right\} \\ & \subseteq \left\{ \sup_{t > 0} \left[|X_t - X_0| - \frac{\gamma}{2} \left(1 + \frac{\mathcal{V}_t}{\mathcal{V}_\tau} \right) \right] < 0 \right\} \end{aligned}$$

Making use of this observation, Bonferroni's inequality, inequalities (17) and (18), we have

$$\Pr \left\{ \sup_{t > 0} \left[|X_t - X_0| - \frac{\gamma}{2} \left(1 + \frac{\mathcal{V}_t}{\mathcal{V}_\tau} \right) \right] < 0 \right\} \geq 1 - 2 \exp \left(-\frac{\gamma^2}{2\mathcal{V}_\tau} \right),$$

from which (19) immediately follows. This completes the proof of the theorem.

K Proof of Corollary 7

Define $g(s) = \sum_{k=2}^{\infty} \frac{s^{k-2}}{k!}$. Then, $g(s) = \frac{e^s - 1 - s}{s^2}$ for $s \neq 0$ and $g(0) = \frac{1}{2}$. It can be shown that $g(s)$ is increasing with respect to s . It is well known that $g(s) \leq \frac{1}{2(1-s/3)}$ for $s \in (0, 3)$.

K.1 Proof of (20)

Making use of the assumptions of the theorem and following the techniques of [2, 6], we have that

$$\begin{aligned}
\mathbb{E} \left[e^{s(X_n - X_{n-1} - a_n)} \mid \mathcal{F}_{n-1} \right] &= \mathbb{E} \left[\sum_{k=0}^{\infty} \frac{s^k}{k!} (X_n - X_{n-1} - a_n)^k \mid \mathcal{F}_{n-1} \right] \\
&= 1 - sa_n + \mathbb{E} \left[\sum_{k=2}^{\infty} \frac{s^k}{k!} (X_n - X_{n-1} - a_n)^k \mid \mathcal{F}_{n-1} \right] \leq 1 - sa_n + s^2 g(sb) \mathbb{E} [(X_n - X_{n-1} - a_n)^2 \mid \mathcal{F}_{n-1}] \\
&= 1 - sa_n + s^2 g(sb) (\mathbb{E} [(X_n - \mathbb{E}[X_n \mid \mathcal{F}_{n-1}])^2 \mid \mathcal{F}_{n-1}] + a_n^2) = 1 - sa_n + s^2 g(sb) [\text{Var}(X_n \mid \mathcal{F}_{n-1}) + a_n^2] \\
&\leq 1 - sa_n + s^2 g(sb) (\sigma_n^2 + a_n^2) \leq \exp(-sa_n + s^2 g(sb) (\sigma_n^2 + a_n^2))
\end{aligned}$$

almost surely. This implies that

$$\mathbb{E}[e^{s(X_n - X_{n-1})} \mid \mathcal{F}_{n-1}] \leq \exp(s^2 g(sb)(\mathcal{V}_n - \mathcal{V}_{n-1})) = \exp\left(\frac{(e^{sb} - 1 - sb)}{b^2}(\mathcal{V}_n - \mathcal{V}_{n-1})\right) \quad (60)$$

almost surely. Thus, we have that $\mathbb{E}[e^{s(X_n - X_{n-1})} \mid \mathcal{F}_{n-1}] \leq \exp((\mathcal{V}_n - \mathcal{V}_{n-1})\varphi(s))$ almost surely, where $\varphi(s) = \frac{e^{sb} - 1 - sb}{b^2}$. Now consider $(X_n)_{n \in \mathbb{N}}$ in the context of Theorem 6. Clearly, $\varphi(0) = \varphi'(0) = 0$ and $\varphi(s)$ is convex. Let $s^* = \frac{\ln(1+b\gamma)}{b}$. By differentiation, it can be shown that $\inf_{s \in (0, \infty)} [\varphi(s) - \gamma s]$ is attained at $s = s^*$. It can be checked that $\varphi(s^*) - \gamma s^* = \frac{\gamma}{b} - \frac{(1+b\gamma)\ln(1+b\gamma)}{b^2}$ and $\frac{\varphi(s^*)}{s^*} = \frac{\gamma}{\ln(1+b\gamma)} - \frac{1}{b}$. Therefore, applying Theorem 6, we have

$$\Pr \left\{ \sup_{n > 0} \left[X_n - X_0 - \gamma \mathcal{V}_m - \left(\frac{\gamma}{\ln(1+b\gamma)} - \frac{1}{b} \right) (\mathcal{V}_n - \mathcal{V}_m) \right] \geq 0 \right\} \leq \left[\exp \left(\frac{\gamma}{b} - \frac{(1+b\gamma)\ln(1+b\gamma)}{b^2} \right) \right]^{\mathcal{V}_m}.$$

This completes the proof of (20).

K.2 Proof of (21)

Since $g(s) \leq \frac{1}{2(1-bs/3)}$, it follows from (60) that

$$\mathbb{E}[e^{s(X_n - X_{n-1})} \mid \mathcal{F}_{n-1}] \leq \exp(s^2 g(sb)(\mathcal{V}_n - \mathcal{V}_{n-1})) \leq \exp\left(\frac{s^2}{2(1-bs/3)}(\mathcal{V}_n - \mathcal{V}_{n-1})\right)$$

almost surely for $s \in (0, \frac{3}{b})$. This implies that for $s \in (0, \frac{3}{b})$, $\mathbb{E}[e^{s(X_n - X_{n-1})} \mid \mathcal{F}_{n-1}] \leq \exp(\varphi(s)(\mathcal{V}_n - \mathcal{V}_{n-1}))$ almost surely, where $\varphi(s) = \frac{s^2}{2(1-bs/3)}$. Applying inequality (4) of Theorem 6 with $s = \frac{1}{\frac{1}{\gamma} + \frac{b}{3}}$ leads to inequality (21).

K.3 Proof of (22)

Applying inequality (60) with $a_i = 0$ and $b = 1$, we have that

$$\mathbb{E}[e^{s(X_n - X_{n-1})} \mid \mathcal{F}_{n-1}] = \mathbb{E}[e^{s(X_n - X_{n-1})}] \leq \exp((e^s - 1 - s)\sigma_n^2) \leq \exp(s^2 \sigma_n^2)$$

almost surely for $s \in (0, \frac{7}{4}]$. Here, we used the fact that $e^s - 1 - s \leq s^2$ for $s \in (0, \frac{7}{4}]$. To see this, consider function $f(s) \stackrel{\text{def}}{=} e^s - 1 - s - s^2$. Note that $f'(s) = e^s - 1 - 2s$ and $f''(s) = e^s - 2$. Clearly, $f'(s)$ decreases from 0 to its minimum at $s = \ln 2$ and then increases for $s \in (\ln 2, \infty)$. Since $f'(0) = 0$, there exists a positive number $\rho > 0$ such that $f'(s)$ is negative for $s \in (0, \rho)$ and positive for $s \in (\rho, \infty)$. Since $f(0) = 0$, it must be true that $f(s)$ decreases from 0 to its minimum as s increases from 0 to ρ and then increases for $s \in (\rho, \infty)$. It follows that if $f(t) < 0$ for some $t > 0$, then $f(s) < 0$ for all $s \in (0, t)$. It can be checked that $f(\frac{7}{4}) < 0$, thus, we have $f(s) < 0$ for all $s \in (0, \frac{7}{4}]$.

So, we have shown that for $s \in (0, \frac{7}{4}]$, $\mathbb{E}[e^{s(X_n - X_{n-1})} \mid \mathcal{F}_{n-1}] \leq \exp((\mathcal{V}_n - \mathcal{V}_{n-1})\varphi(s))$ almost surely, where $\varphi(s) = s^2$. Consider $(X_n)_{n \in \mathbb{N}}$ in the context of Theorem 6. Let $\varepsilon > 0$. Note that $\frac{d\varphi(s)}{ds} = \varepsilon$ if $s = \frac{\varepsilon}{2}$. Accordingly, $\beta(\varepsilon) = \frac{\varepsilon}{2}$ and $s(\beta(\varepsilon) - \varepsilon) = -\frac{\varepsilon^2}{4}$. Thus, applying Theorem 6, we have

$$\Pr \left\{ \sup_{n>0} \left[X_n - X_0 - \varepsilon \mathcal{V}_m - \frac{\varepsilon}{2} (\mathcal{V}_n - \mathcal{V}_m) \right] \geq 0 \right\} \leq \exp \left(-\frac{\varepsilon^2}{4} \right)^{\mathcal{V}_m}$$

for $\varepsilon \leq \frac{7}{2}$. Letting $\varepsilon \mathcal{V}_m = \gamma$ in the above inequality leads to (22). This completes the proof of the corollary.

L Proof of Corollary 8

Define $Y_t = X_t - \lambda t$ for $t \in \mathbb{R}^+$. Let $\mathcal{V}_t = t$ for $t \in \mathbb{R}^+$ and let $\varphi(s) = \lambda(e^s - 1 - s)$ for $s \in \mathbb{R}$. Consider the process $(Y_t)_{t \in \mathbb{R}^+}$ in the context of Theorem 6. Clearly, $(Y_t)_{t \in \mathbb{R}^+}$ is actually a right-continuous stochastic process such that $\mathbb{E}[\exp(s(Y_{t'} - Y_t)) \mid \mathcal{F}_t] \leq \exp((\mathcal{V}_{t'} - \mathcal{V}_t)\varphi(s))$ almost surely for arbitrary $t' \geq t \geq 0$ and $s \in (-\infty, \infty)$. Since the derivative of $\varphi(s) - \gamma s$ with respect to s is $\lambda e^s - \lambda - \gamma$, it follows that $\inf_{s \in (0, \infty)} [\varphi(s) - \gamma s]$ is attained at $s^* = \ln \frac{\lambda + \gamma}{\lambda}$. Consequently,

$$\inf_{s \in (0, \infty)} [\varphi(s) - \gamma s] = \varphi(s^*) - \gamma s^* = \gamma - (\lambda + \gamma) \ln \frac{\lambda + \gamma}{\lambda} \quad \text{and} \quad \beta(\gamma) = \frac{\varphi(s^*)}{s^*} = \frac{\gamma}{\ln \frac{\lambda + \gamma}{\lambda}} - \lambda.$$

Applying Theorem 6 gives

$$\Pr \left\{ \sup_{t>0} \left[Y_t - \gamma t - \left(\frac{\gamma}{\ln \frac{\lambda + \gamma}{\lambda}} - \lambda \right) (t - \tau) \right] \geq 0 \right\} \leq \left[\exp \left(\gamma - (\lambda + \gamma) \ln \frac{\lambda + \gamma}{\lambda} \right) \right]^\tau,$$

which implies that $\Pr\{\sup_{t>0} [X_t - (\lambda + \gamma)\tau - \frac{\gamma(t-\tau)}{\ln(1+\frac{\gamma}{\lambda})}] \geq 0\} \leq [(\frac{\lambda}{\lambda+\gamma})^{\lambda+\gamma} e^\gamma]^\tau$ for any $\tau > 0$ and $\gamma > 0$.

On the other hand, since the derivative of $\varphi(-s) - \gamma s$ with respect to s is $-\lambda e^{-s} + \lambda - \gamma$, it follows that $\inf_{s \in (0, \infty)} [\varphi(-s) - \gamma s]$ is attained at $s^* = \ln \frac{\lambda}{\lambda - \gamma}$. Consequently,

$$\inf_{s \in (0, \infty)} [\varphi(-s) - \gamma s] = -\gamma - (\lambda - \gamma) \ln \frac{\lambda}{\lambda - \gamma} \quad \text{and} \quad \alpha(\gamma) = \frac{\varphi(-s^*)}{s^*} = \frac{\gamma}{\ln \frac{\lambda}{\lambda - \gamma}} + \lambda.$$

Applying Theorem 6 yields

$$\Pr \left\{ \inf_{t>0} \left[Y_t + \gamma t + \left(\frac{\gamma}{\ln \frac{\lambda}{\lambda - \gamma}} + \lambda \right) (t - \tau) \right] \leq 0 \right\} \leq \left[\exp \left(-\gamma - (\lambda - \gamma) \ln \frac{\lambda}{\lambda - \gamma} \right) \right]^\tau,$$

which implies that $\Pr\{\inf_{t>0} [X_t - (\lambda - \gamma)\tau + \frac{\gamma(t-\tau)}{\ln(1-\frac{\gamma}{\lambda})}] \leq 0\} \leq [(\frac{\lambda}{\lambda-\gamma})^{\lambda-\gamma} e^{-\gamma}]^\tau$ for any $\tau > 0$ and $\gamma > 0$. This completes the proof of the corollary.

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